2011

Symmetric presentation of finite groups

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Symmetric Presentation of Finite Groups

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

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Abstract

The main goal of this project is to construct finite homomorphic image of monomial infinite semi-direct products which are called progenitors. The progenitor can be written as $m^{*n} :_m N$, where N is the control group and m^{*m} is a free product of n copies of the cyclic group C_m of order m. In this thesis, we will restrict our attention to the cases that $m = 2$ and $m = 3$. We will use the technique of double coset enumeration to construct the finite groups $S_4 \times 2$, $S_5 \times 2$, A_7, $A_7 \times S_3$, $3{}^{\cdot}A_7$ and $M_{12} : 2$ as the homomorphic images of the progenitors $2^{*3} :_m S_3$, $2^{*4} :_m S_4$, $3^{*5} :_m S_5$, $3^{*5} :_m S_5$, $3^{*3} :_m S_4$ and $3^{*5} :_m S_5$ respectively.

ACKNOWLEDGEMENTS

I want to thank all of the following professors for guidance and support during my MA program. Without your help, I would not have been able to complete my degree. I would like to thank **Dr. Zahid Hasan** and highly appreciative of your guidance and support with the thesis. I would like to thank **Dr. Joseph Chavez** for giving me an opportunity to enter this MA program. I would like to thank **Dr. John Sarli** and **Dr. Corey Dunn** for putting in your precious time to review my thesis. I also want to thank **Dr. Charles Stanton** for responding to all my concerns and questions during my research. Once again I want to thank you all for help and support during my research. From the bottom of my heart, I am extremely appreciative to all of you and will never forget any of you.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

Matrices and permutations, the two common methods for working with groups tend to be problematic or even uncontrollable for large finite groups, in particular larger sporadic groups. Matrix multiplication for large matrices is very time consuming. The basic element information is not easily recovered using matrices and matrix operation are time consuming. While basic operation on permutations are straight forward, their length is inconvenient.

In this thesis, we provide an alternative convenient and efficient method. This method can be applied to many groups, including all finite non-abelian simple groups, see [2]. Several sporadic simple groups can be constructed using this approach, see for example [7]. In this thesis, we will use this approach to construct several finite groups including $S_4 \times 2$, $S_5 \times 2$, A_7, $A_7 \times S_3$, $3{\cdot}A_7$ and $M_{12} : 2$ as homomorphic images of $2^{*3} :_m S_3$, $2^{*4} :_m S_4$, $3^{*5} :_m S_5$, $3^{*5} :_m S_5$, $3^{*3} :_m S_4$ and $3^{*5} :_m S_5$ respectively. The technique of double coset enumeration not only enables us to express elements of a group in a much more concise manner, but this technique also allows the multiplication and inversion of elements represented in short form, in a straight forward manner.

Symmetric generation of a group

We now give the basic definitions that we will use throughout this paper.

Let G be a group and $T = \{t_1, t_2, \ldots, t_n\}$ be a subset of G, and let

$$\overline{T} = \{<\bar{t_i}> | 1 \le i \le n\}.$$

If

(i) $G = \langle T \rangle$, and

(ii) The set normalizer N of $\overline{T}$ in G acts on $\overline{T}$ transitively,

then G is a homomorphic image of the *progenitor* $m^{*n} : N$ where m^{*n} is a free product of n cyclic groups of order m. In this thesis, we will restrict our attention to the cases that $m = 2$ and $m = 3$, where $|t_i| = m$, and N is a group of automorphisms of m^{*n} which acts on the n cyclic subgroups by conjugation. If π is an element of N, then $t_i^{\pi} = \pi t_i^{\pi}$. Every element of $m^{*n} : N$ can be written, not uniquely, as πw, where $\pi \in N$ and w is a word in $t_i s$. We factor the progenitors by relations of the form nw, where $n \in N$ and w is a word in the $t_i s$. We write this as

$$G = \frac{m^{*n} : N}{nw}.$$

Double coset enumeration

We factor $2^{*n} : N$ by appropriate relators and construct finite homomorphic images. In order to construct

$$G = \frac{2^{*n}:N}{\pi_1 w_1, \pi_2 w_2, \pi_3 w_3 \ldots, \pi_s w_s},$$

we perform the manual double coset enumeration of G over N.

We let i to represent t_i and also represent the double coset Nt_i by i. If $Nt_i t_j = Nt_l t_k$, then we write it as $ij \sim lk$. The double coset

$$NwN = \{Nw^n | n \in N\}$$

is written as $[w]$. The point stabilizer of the point i is denoted by N^i and the coset stabilizer of the double coset Nw is given by

$$N^{(w)} = \{n \in N | Nw^n = Nw\}.$$

We will frequently use the formula for the number of single cosets in the double coset NwN, namely

$$NwN = \frac{|N|}{|N^{(w)}|}.$$

We need to find all the double cosets and the numbers of single cosets in each double coset. The two chapters (Chapter-2 and Chapter-3), are written in detail manner to facilitate a better reading of the rest of the thesis.

Chapter 2

Construction of $S_4 \times 2$

Factoring the progenitor $2^{*3} : S_3$ by the relation $t_0t_1 = xt_0t_2$, we obtain the finite image

$$G \cong \frac{2^{*3} :_m S_3}{t_0t_1=(0,1,2)t_0t_2}$$

The symmetric presentation of the progenitor $2^{*3} : S_3$ can be given by:

$$2^{*3} : S_3 \cong < x, y, t | x^3 = y^2 = (xy)^2, (t, y), t_0t_1 = (0, 1, 2)t_0t_2 >.$$

The control group is $N = S_3 \cong < s, y | x^3 = y^2 = (xy)^2 >$. The action on the symmetric generators is given by $x \sim (0, 1, 2)$, $y \sim (1, 2)$, and our relation is $t_0t_1 = (0, 1, 2)t_0t_2 \sim [(0, 1, 2)t_0]^4$.

Double coset enumeration of G over S_3.

We note that,

$$NeN = \{Nen|n \in N\} = \{Nn|n \in N\} = \{\text{N}\}.$$

Let $[*]$ represents for the double coset NeN. $[*]$ has only one distinct single coset, namely N. Since, $N \cong S_3$ acts transitively on $\{0, 1, 2\}$. Therefore, N has one orbit namely $\{0, 1, 2\}$. In $\{0, 1, 2\}$, we take one representative 0 from the orbit and find which double coset Nt_0 belongs.

Clearly, $Nt_0 \in Nt_0N = \{Nt_0^n | n \in N\} = \{Nt_0, Nt_1, Nt_2\}$, and denote Nt_0N by $[0]$. Now, consider the coset stabilizer $N^{(0)}$. The coset stabilizer $N^{(0)}$ is equal to the point stabilizer

N^0 and is given by:

$N^{(0)} = N^0 =< e, (1,2) >$.

Therefore, the number of single cosets in [0] are

$$\frac{|N|}{|N^{(0)}|} = \frac{3!}{2} = \frac{6}{2} = 3.$$

The orbits of $N^{(0)}$ on $\{0, 1, 2\}$ are $\{0\}$ and $\{1, 2\}$. Now, take one representative from each orbit, namely 0 and 1, and find the double coset to which Nt_0t_0 and Nt_0t_1 belong. However,

$Nt_0t_0 = N \in [*]$

$Nt_0t_1 = Nt_0t_1 \in Nt_0t_1N$ (New double coset denoted by [01])

Therefore, one symmetric generator goes back to the double coset $\{*\}$ and two extend.

Next, we consider the double coset [01].

$N^{(01)} \geq N^{01} =< e >$. By our relation $t_0t_1 = (0,1,2)t_0t_2$, $\Rightarrow Nt_0{t_1}^{(1,2)} = Nt_0t_2$

$\Rightarrow (1,2) \in N^{(01)}$, and our relation gives $01 \sim 02$.

$\Rightarrow N^{(01)} =< e, (1,2) >= \{e, (1,2)\}$.

So, the number of single cosets in the double coset [01] are at most

$$\frac{|N|}{|N^{(01)}|} = \frac{3!}{2} = \frac{6}{2} = 3.$$ Each single coset has two different names.

We can identify the three single cosets in [01] by conjugating $(01 \sim 02)$ by the transversal of $N^{(01)}$ in N.

$T = \{e, (0,1,2), (0,2,1)\}$

$(01 \sim 02)^{(0,1,2)} = (12 \sim 10)$

$(01 \sim 02)^{(0,2,1)} = (20 \sim 21)$

Hence, the three distinct right single cosets are:

$01 \sim 02, 10 \sim 12, 20 \sim 21$.

Since the orbits of $N^{(01)}$ on $\{0, 1, 2\}$ are $\{0\}$ and $\{1, 2\}$. We take representatives 0 and 1 from the two orbits, and determine which double coset $Nt_0t_1t_1$ and $Nt_0t_1t_0$ belong? However,

$Nt_0t_1t_1 = Nt_0 \in [0]$

$Nt_0t_1t_0 \in Nt_0t_1t_0N$. (denote this new double coset by [010]).

Therefore, two symmetry generators go back and one advances.

Now, let us consider the double coset [010].

$N^{(010)} \geq N^{010} =< e >$, and from our relation $01 \sim 02$, we have $010 \sim 020$.
$Nt_0t_1t_0{}^{(1,2)} = Nt_0t_2t_0 = Nt_0t_1t_0 \Rightarrow (1,2) \in N^{(010)}$.
$Nt_1t_0t_1{}^{(0,2)} = Nt_1t_2t_1 = Nt_1t_0t_1 \Rightarrow (0,2) \in N^{(010)}$.
$\Rightarrow N^{(010)} =< (0,2),(1,2) >= \{e,(0,1),(0,2),(1,2),(0,1,2),(0,2,1)\} \cong S_3$.
Hence, the number of single coset in [010] is at most

$$\frac{|N|}{|N^{(010)}|} = \frac{3!}{3!} = \frac{6}{6} = 1.$$

The single coset in this double coset has six different names. We now find the six names of the single coset $Nt_0t_1t_0$
By our relation $t_0t_1 = (0,1,2)t_0t_2$ and $(01 \sim 02,\ 10 \sim 12,\ 20 \sim 21)$ above, we have the following:

- $t_0t_1t_0 = (0,1,2)t_0t_2t_0 \Rightarrow 010 = (0,1,2)020$
- $t_0t_1{}^{(0,2,1)} = (0,1,2)t_0t_2{}^{(0,2,1)}$
  $t_2t_0 = (2,0,1)t_2t_1$
  Now, $t_0t_1t_0 = (0,1,2)t_0t_2t_0 = (0,1,2)t_0(2,0,1)t_2t_1 = (0,2,1)t_1t_2t_1$
  $\Rightarrow 010 = (0,2,1)121$.
- But $12 \sim 10$ and $010 = (0,2,1)121 \Rightarrow 010 = (0,2,1)101$
- $t_0t_1{}^{(0,1,2)} = (0,1,2)t_0t_2{}^{(0,1,2)}$
  $t_1t_2 = (0,1,2)t_1t_0$
  Also $t_0t_1t_0 = (0,1,2)t_0t_2t_0$
  $\Rightarrow t_0t_2t_0 = (0,2,1)t_0t_1t_0 = (0,2,1)t_0(0,2,1)t_1t_2 = (0,2,1)(0,2,1)t_2t_1t_2$
  $= (0,1,2)t_2t_1t_2 \Rightarrow 010 = (0,1,2)212$
- But $20 \sim 21$
  So, $t_0t_1t_0 = (0,1,2)t_2t_1t_2 = (0,1,2)t_2t_0t_2 \Rightarrow 010 = (0,1,2)202$

So, all the names of the right single coset in double coset [010] are:
$010 \sim 020 \sim 121 \sim 101 \sim 212 \sim 202$.

$N^{(010)} =< (0,2),(1,2) >$ acts transitively on $\{0,1,2\}$. Thus, $N^{(010)}$ has only a single orbit $\{0,1,2\}$ on $\{0,1,2\}$. Now take one representative from the orbit namely 0, and determine which double coset $Nt_0t_1t_0t_0$ belongs. However,
$Nt_0t_1t_0t_0 = Nt_0t_1 \in [01]$.
Therefore, all symmetric generators go back to the double coset [01] and none of them advance. Hence, we must have completed the double coset enumeration. The double coset enumeration shows that the index of $N \cong S_3$ in G is at most

$$|G| \leq (1+3+3+1)|N| = (8)6 = 48.$$

All the information is summarized in the Cayley diagram below.

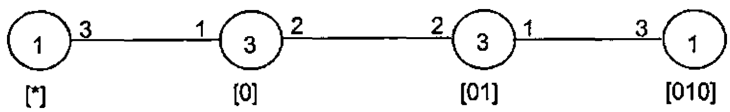

Figure 2.1: The Cayley diagram of $S_4 \times 2$ over S_4.

Permutation Representation.

Cosets	$x \sim (0,1,2)$	$y \sim (1,2)$	t_0
$1.N$	$1.N$	$1.N$	$2.Nt_0$
$2.Nt_0$	$3.Nt_1$	$2.Nt_0$	$1.N$
$3.Nt_1$	$4.Nt_2$	$4.Nt_2$	$6.Nt_1t_0$
$4.Nt_2$	$2.Nt_0$	$3.Nt_1$	$7.Nt_2t_0$
$5.Nt_0t_1$	$6.Nt_1t_0$	$5.Nt_0t_1$	$8.Nt_0t_1t_0$
$6.Nt_1t_0$	$7.Nt_2t_0$	$7.Nt_2t_0$	$3.Nt_1$
$7.Nt_2t_0$	$5.Nt_0t_1$	$6.Nt_1t_0$	$4.Nt_2$
$8.Nt_0t_1t_0$	$8.Nt_0t_1t_0$	$8.Nt_0t_1t_0$	$5.Nt_0t_1$

Table 2.1: Permutation Representation of $S_4 \times 2$

The homomorphic image of G.

Define $\widehat{\alpha} : G \to S_8$, where (8) is a number of single cosets from the Cayley diagram above gives the action of G on the 8 cosets. From the labeling given in Table-2.1, we compute the actions of x, y and t_0 on the 8 cosets.

$\widehat{\alpha}(x) = (2,3,4)(5,6,7)$.

$\widehat{\alpha}(y) = (3,4)(6,7)$.

$\widehat{\alpha}(t_0) = (1,2)(3,6)(4,7)(5,8)$.

We now verity that $\widehat{\alpha}$ is a homomorphism.

1. We want to show that $\widehat{\alpha}(t_0)$ has exactly 3 conjugates under conjugation by $< \widehat{\alpha}(x), \widehat{\alpha}(y) >$.
   $\widehat{\alpha}(t_0)^{\widehat{\alpha}(x)} = (1,3)(4,7)(2,5)(6,8) = \widehat{\alpha}(t_1)$.
   $\widehat{\alpha}(t_1)^{\widehat{\alpha}(x)} = (1,4)(2,5)(3,6)(7,8) = \widehat{\alpha}(t_2)$.
   $\widehat{\alpha}(t_2)^{\widehat{\alpha}(x)} = (1,2)(3,6)(4,7)(5,8) = \widehat{\alpha}(t_0)$.
   $\widehat{\alpha}(t_0)^{\widehat{\alpha}(y)} = (1,2)(3,6)(4,7)(5,8) = \widehat{\alpha}(t_0)$.
   $\widehat{\alpha}(t_1)^{\widehat{\alpha}(y)} = (1,4)(2,5)(3,6)(7,8) = \widehat{\alpha}(t_2)$.
   $\widehat{\alpha}(t_2)^{\widehat{\alpha}(y)} = (1,3)(4,7)(2,5)(6,8) = \widehat{\alpha}(t_1)$.

2. $\widehat{\alpha}(N)$ acts as N on $\{\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(t_2)\}$ by conjugation; we note that $\widehat{\alpha}(x) : (\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(t_2))$, $\widehat{\alpha}(y) : (\widehat{\alpha}(t_1), \widehat{\alpha}(t_2))$, and $< \widehat{\alpha}(x), \widehat{\alpha}(y) > \cong N = S_3$.
   Then, $\widehat{\alpha}(G) = \widehat{\alpha}(2^{*3} : S_3) = \widehat{\alpha} < N, t_0 > = < \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$ is a homomorphic image of $(2^{*3} : S_3)$.

Next, we want the homomorphic image of $G = \frac{2^{*3}:S_3}{t_0t_1=(0,1,2)t_0t_2}$.
Now, $<\widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0)> \leq S_8$ is a homomorphic image of $G = \frac{2^{*3}:S_3}{t_0t_1=(0,1,2)t_0t_2}$ if the additional relation $t_0t_1 = (0,1,2)t_0t_2$ holds in $<\widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0)>$.
Thus, we want to show that $\widehat{\alpha}(t_0t_1) = \widehat{\alpha}((0,1,2)t_0t_2)$
$\widehat{\alpha}(t_0t_1) = \widehat{\alpha}(t_0)\widehat{\alpha}(t_1) = (1,2)(3,6)(4,7)(5,8)(1,3)(4,7)(2,5)(6,8) = (1,5,6)(2,3,8)$.
$\widehat{\alpha}((0,1,2)t_0t_2) = \widehat{\alpha}((0,1,2))\widehat{\alpha}(t_0)\widehat{\alpha}(t_2) = (2,3,4)(5,6,7)(1,2)(3,6)(4,7)$
$(5,8)(1,4)(2,5)(3,6)(7,8) = (1,5,6)(2,3,8)$.

Therefore, $\widehat{\alpha} : G \to S_8$ is a homomorphism map.
Now, by First Isomorphism Theorem, we have
$G|ker(\widehat{\alpha}) \cong im(\widehat{\alpha})$
$\Rightarrow \frac{|G|}{|ker(\widehat{\alpha})|} = |im(\widehat{\alpha})| = |<\widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0)>| = 48$
$\Rightarrow |G| = 48 \times |ker(\widehat{\alpha})|$, where $(|ker(\widehat{\alpha})| \geq 1)$.
$\Rightarrow |G| \geq 48$.
But early we found that $|G| \leq 48$. Therefore, we can conclude that $|G| = 48$. Since $|G| = 48$
$\Rightarrow |ker(\widehat{\alpha})| = 1$.
$\Rightarrow \widehat{\alpha}$ is also an isomorphism.
$\Rightarrow G \cong \widehat{\alpha}(G) = <\widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0)>$.

It is clear from the Cayley diagram of G over S_3 that every element of G can be written as πw, where π is a permutation of S_3 on three letters, and w is a word in the t_{i_s} of the length at most three. We call this representation of elements of G the symmetric representation. We now give examples of elements of G in both symmetric and permutation representation.

Permutation Representation to Symmetric Presentation

Let $p = (1,6,7)(3,4,8)$ be an element in the permutation representation.
We know that $Np = Nw$.
Since, $p \in Np \Rightarrow p \in Nw$.
We know that $Np = 1^p$
$\Rightarrow Np = g(1) = 6 = Nt_1t_0$.
$\Rightarrow p = nw$, where $n \in N$.

To represent p in the symmetric representation form, we must find $n \in N \cong S_3$.

We have $p(1) = 6$ and Table-2.1 gives that Nt_1t_0 is labeling 6.

If $p = nw$.

$\Rightarrow n = pw^{-1}$

$= p(t_1t_0)^{-1}$

$= p\bar{t_0}\bar{t_1}$

$= p\widehat{\alpha}(t_0)\widehat{\alpha}(t_1)$

$= (1,6,7)(3,4,8)(1,2)(3,6)(4,7)(5,8)(1,3)(4,7)(2,5)(6,8)$

$= (2,3,4)(5,6,7)$.

From our Table-2.1 again,

$n = (0,1,2)$.

Therefore, p can be written in the symmetric representation as follows

$p = (0,1,2)t_1t_0$.

Symmetric Presentation to Permutation Representation

Let $(0,1,2)t_1t_0$ be an element in the symmetric representation.

$\Rightarrow p = \widehat{\alpha}(0,1,2)\widehat{\alpha}(t_1)\widehat{\alpha}(t_0) = (2,3,4)(5,6,7)(1,3)(4,7)(2,5)(6,8)(1,2)(3,6)(4,7)(5,8)$

$= (1,6,7)(3,4,8)$.

Therefore, $(0,1,2)t_1t_0$ can be written in the permutation representation as follow

$(0,1,2)t_1t_0 = (1,6,7)(3,4,8)$.

Chapter 3

Construction of $S_5 \times 2$

Factoring the progenitor $2^{*4} : S_4$ by the relation $t_0t_1 = (0,1,2)t_0t_2$, we obtain the finite image

$$G \cong \frac{2^{*4} :_m S_4}{t_0t_1=(0,1,2)t_0t_2}$$

The symmetric presentation of the progenitor $2^{*4} :_m S_4$ can be given by:

$$2^{*4} :_m S_4 \cong < x, y, t | x^4, y^2, (xy)^3, t^2, (t, xy), t_0t_1 = (0,1,2)t_0t_2 >$$

The control group is $N = S_4 \cong < x, y | x^3 = y^2 = (xy)^3 >$. The action on the symmetric generators is given by $x \sim (0,1,2,3)$, $y \sim (0,1)$, and our relation is $t_0t_1 = (0,1,2)t_0t_2$.

Double coset enumeration of G over S_4.

We note that,

$$NeN = \{Nen | n \in N\} = \{Nn | n \in N\} = \{\text{N}\}.$$

Let [*] represents the double coset NeN. [*] has only one distinct single coset, namely N. Since, $N \cong S_4$, acts transitively on $\{0,1,2,3\}$. Therefore, N has one orbit namely $\{0,1,2,3\}$. In $\{0,1,2,3\}$ we take one representative 0 from the orbit and find which double coset Nt_0 belongs.

Clearly, $Nt_0 \in Nt_0N = \{Nt_0^n | n \in N\} = \{Nt_0, Nt_1, Nt_2, Nt_3\}$, and denote Nt_0N by [0]. Now, consider the coset stabilizer $N^{(0)}$. The coset stabilizer $N^{(0)}$ is equal to the point stabilizer N^0 and is given by:

$$N^{(0)} = N^0 =< (1,2,3), (1,2) >\cong S_3.$$

Therefore, the number of single cosets in [0] are

$$\frac{|N|}{|N^{(0)}|} = \frac{4!}{3!} = \frac{24}{6} = 4.$$

The orbits of $N^{(0)}$ on $\{0, 1, 2, 3\}$ are $\{0\}$ and $\{1, 2, 3\}$. Now, take one representative from each orbit, namely 0 and 1, and find the double coset to which Nt_0t_0 and Nt_0t_1 belong. However,

$Nt_0t_0 = N \in [*]$

$Nt_0t_1 = Nt_0t_1 \in Nt_0t_1N$ (New double coset denoted by [01])

Therefore, one symmetric generator goes back and three extend.

Next, we consider the double coset [01].

We know that $N^{(01)} \geq N^{01} =< e, (2,3) >$, and by our relation $t_0t_1 = (0,1,2)t_0t_2$ we have,

$Nt_0{t_1}^{(1,2,3)} = Nt_0t_2 \Rightarrow (1,2,3) \in N^{(01)}$, our relation gives $01 \sim 02$.

$N^{(01)} =< (1,2,3), (2,3) >= \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\} \cong S_3$.

Therefore, the number of single right coset in the double cosets NNt_0t_1N are at most

$$\frac{|N|}{|N^{(01)}|} = \frac{4!}{3!} = \frac{24}{6} = 4.$$ Each single coset has three different names.

And now, by conjugating (01) with elements of $N^{(01)}$, we obtain the following relation: $(01 \sim 02 \sim 03)$.

We can identify all the four single cosets in [01] by conjugating $(01 \sim 02 \sim 03)$ with elements of the transversal of $N^{(01)}$ in N.

$T = \{e, (0,1), (0,2), (0,3)\}$

$(01 \sim 02 \sim 03)^{(0,1)} = (10 \sim 12 \sim 13)$

$(01 \sim 02 \sim 03)^{(0,2)} = (21 \sim 20 \sim 23)$

$(01 \sim 02 \sim 03)^{(0,3)} = (31 \sim 32 \sim 30)$

Hence, the four distinct right single cosets are:

$01 \sim 02 \sim 03, 10 \sim 12 \sim 13, 20 \sim 21 \sim 23, 30 \sim 31 \sim 32$

Since the orbits of $N^{(01)}$ on $\{0,1,2,3\}$ are $\{0\}$ and $\{1,2,3\}$. We take representatives 0 and 1 from the orbits, and determine which double coset $Nt_0t_1t_0$ and $Nt_0t_1t_1$ belong. However,

$Nt_0t_1t_1 = Nt_0 \in [0]$

$Nt_0t_1t_0 \in Nt_0t_1t_0N$. Denoted this new double coset by [010].

Therefore, three symmetric generators go back to the double coset [0], and one advances.

Now, let us consider the double coset [010].

$N^{(010)} \geq N^{010} =< e, (2,3) >$, and from our relation $01 \sim 02 \sim 03$, we have $010 \sim 020 \sim 030$.

$Nt_0t_1t_0{}^{(1,2)} = Nt_0t_2t_0 = Nt_0t_1t_0 \Rightarrow (1,2) \in N^{(010)}$.

$Nt_3t_0t_3{}^{(0,1,2)} = Nt_3t_1t_3 = Nt_3t_0t_3 \Rightarrow (0,1,2) \in N^{(010)}$.

$\Rightarrow N^{(010)} \geq< (0,1,2), (1,2), (2,3) >\cong S_4$.

Hence, the number of right single coset in [010] is at most

$$\frac{|N|}{|N^{(010)}|} = \frac{4!}{4!} = \frac{24}{24} = 1.$$

The single cosets in this double coset has twelve different names. We now find the twelve names of the single coset $Nt_0t_1t_0$.

By our relation $t_0t_1 = (0,1,2)t_0t_2$ and above relations, then we obtain the following relations:

- $t_0t_1{}^{(0,1,2)} = (0,1,2)t_0t_2{}^{(0,1,2)} \Rightarrow t_2t_0 = (0,1,2)t_2t_1$
  $\Rightarrow t_0t_1t_0 = (0,1,2)t_0t_2t_0 = (0,1,2)t_0(0,1,2)t_2t_1 = (0,2,1)t_1t_2t_1$
  $\Rightarrow 010 = (0,2,1)121$
- But $12 \sim 10 \Rightarrow 010 = (0,2,1)121 = (0,2,1)101$
- $t_0t_1{}^{(0,3,1)} = (0,1,2)t_0t_2{}^{(0,3,1)} \Rightarrow t_3t_0 = (3,0,2)t_3t_2$
  $\Rightarrow t_0t_1t_0 = (0,1,2)t_0t_3t_0 = (0,1,2)t_0(3,0,2)t_3t_2 = (0,1,3)t_2t_3t_2$
  $\Rightarrow 010 = (0,1,3)232$

- $t_0t_1{}^{(0,2,1,3)} = (0,1,2)t_0t_2{}^{(0,2,1,3)} \Rightarrow t_2t_3 = (1,2,3)t_2t_1$
  $\Rightarrow t_0t_1t_0 = (0,1,3)t_2t_3t_2 = (0,1,3)(1,2,3)t_2t_1t_2 = (0,2,3)t_2t_1t_2$
  $\Rightarrow 010 = (0,2,3)212$

- Since $t_2t_0 = (2,0,1)t_2t_1 \Rightarrow t_2t_1 = (2,1,0)t_2t_0$
  $\Rightarrow t_0t_1t_0 = (0,2,3)t_2t_1t_2 = (0,2,3)(2,1,0)t_2t_0t_2 = (0,1)(2,3)t_2t_3t_2$
  $\Rightarrow 010 = (0,1)(2,3)202$

- $t_0t_1{}^{(0,1,3)} = (0,1,2)t_0t_2{}^{(0,1,3)} \Rightarrow t_1t_3 = (1,3,2)t_1t_2$
  $\Rightarrow t_0t_1t_0 = (0,2,1)t_1t_2t_1 = (0,2,1)(1,2,3)t_1t_3t_1 = (0,3,1)t_1t_3t_1$
  $\Rightarrow 010 = (0,3,1)131$

- $t_0t_3 = (1,3,2)t_0t_2$
  $\Rightarrow t_0t_1t_0 = (0,1)(2,3)t_2t_0t_2 = (0,1)(2,3)t_2(1,2,3)t_0t_3 = (0,2,1)t_3t_0t_3$
  $\Rightarrow 010 = (0,2,1)303$

- But $30 \sim 31 \sim 32$ and $010 \sim 030$
  $\Rightarrow 010 \sim 303 \sim 313 \sim 323$

So, all the names of the single coset in double coset [010] are:

$$010 \sim 020 \sim 030 \sim 101 \sim 121 \sim 131 \sim 202 \sim 212 \sim 232 \sim 303 \sim 313 \sim 323$$

Since $N^{(010)}$ acts transitively on $\{0,1,2,3\}$, $N^{(010)}$ has a single orbit $\{0,1,2,3\}$. Now take representative 0, and determine which double coset $Nt_0t_1t_0t_0$ belongs. However, $Nt_0t_1t_0t_0 = Nt_0t_1 \in [01]$

Therefore, all symmetric generators go back to the double coset [01] and none of them advances. Hence, we must have completed the double coset enumeration. All the information is summarized in the Cayley diagram below.

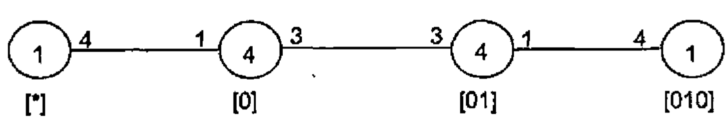

Figure 3.1: The Cayley diagram of $S_5 \times 2$ over S_4.

Permutation Representation.

Cosets	$x \sim (0,1,2,3)$	$y \sim (0,1)$	$(0,1,2)$	t_0
$1.N$	$1.N$	$1.N$	$1.N$	$2.Nt_0$
$2.Nt_0$	$3.Nt_1$	$3.Nt_1$	$3.Nt_1$	$1.N$
$3.Nt_1$	$4.Nt_2$	$2.Nt_0$	$4.Nt_2$	$7.Nt_1t_0$
$4.Nt_2$	$5.Nt_3$	$4.Nt_2$	$2.Nt_0$	$8.Nt_2t_0$
$5.Nt_3$	$2.Nt_0$	$5.Nt_3$	$5.Nt_3$	$9.Nt_3t_0$
$6.Nt_0t_1$	$7.Nt_1t_0$	$7.Nt_1t_0$	$7.Nt_1t_0$	$10.Nt_0t_1t_0$
$7.Nt_1t_0$	$8.Nt_2t_0$	$6.Nt_0t_1$	$8.Nt_2t_0$	$3.Nt_1$
$8.Nt_2t_0$	$9.Nt_3t_0$	$8.Nt_2t_0$	$6.Nt_0t_1$	$4.Nt_2$
$9.Nt_3t_0$	$6.Nt_0t_1$	$9.Nt_3t_0$	$9.Nt_3t_0$	$5.Nt_3$
$10.Nt_0t_1t_0$	$10.Nt_0t_1t_0$	$10.Nt_0t_1t_0$	$10.Nt_0t_1t_0$	$6.Nt_0t_1$

Table 3.1: Permutation Representation of $S_5 \times 2$

The double coset enumeration shows that the index of $N \cong S_4$ in G is at most

$$|G| \leq (1+4+4+1)|N| = (10)24 = 240.$$

The homomorphic image of G.

Define $\widehat{\alpha} : G \to S_{10}$, where (10) is a number of single cosets from the Cayley diagram above gives the action of G on the 10 cosets. From the labeling given in Table-3.1, we compute the actions of x, y and t_0 on the 10 cosets.
$\widehat{\alpha}(x) = (2,3,4,5)(6,7,8,9)$.
$\widehat{\alpha}(y) = (2,3)(6,7)$.
$\widehat{\alpha}(t_0) = (1,2)(3,7)(4,8)(5,9)(6,10)$.
We now verify that $\widehat{\alpha}$ is a homomorphism.

1. We want to show that $\widehat{\alpha}(t_0)$ has exactly 4 conjugates under conjugation by $< \widehat{\alpha}(x), \widehat{\alpha}(y) >$.
   $\widehat{\alpha}(t_0)^{\widehat{\alpha}(x)} = (1,3)(4,8)(5,9)(2,6)(7,10) = \widehat{\alpha}(t_1)$.
   $\widehat{\alpha}(t_1)^{\widehat{\alpha}(x)} = (1,4)(5,9)(2,6)(3,7)(8,10) = \widehat{\alpha}(t_2)$.
   $\widehat{\alpha}(t_2)^{\widehat{\alpha}(x)} = (1,5)(2,6)(3,7)(4,8)(9,10) = \widehat{\alpha}(t_3)$.
   $\widehat{\alpha}(t_3)^{\widehat{\alpha}(x)} = (1,2)(3,7)(4,8)(5,9)(6,10) = \widehat{\alpha}(t_0)$.
   $\widehat{\alpha}(t_0)^{\widehat{\alpha}(y)} = (1,3)(4,8)(5,9)(2,6)(7,10) = \widehat{\alpha}(t_1)$.
   $\widehat{\alpha}(t_1)^{\widehat{\alpha}(y)} = (1,2)(3,7)(4,8)(5,9)(6,10) = \widehat{\alpha}(t_0)$.

$\widehat{\alpha}(t_2)^{\widehat{\alpha}(y)} = (1,4)(5,9)(2,6)(3,7)(8,10) = \widehat{\alpha}(t_2)$.
$\widehat{\alpha}(t_3)^{\widehat{\alpha}(y)} = (1,5)(2,6)(3,7)(4,8)(9,10) = \widehat{\alpha}(t_3)$.

2. $\widehat{\alpha}(N)$ acts as N on $\{\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(t_2), \widehat{\alpha}(t_3)\}$ by conjugation,we note that,
   $\widehat{\alpha}(x) = (\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(t_2), \widehat{\alpha}(t_3))$,
   $\widehat{\alpha}(y) = (\widehat{\alpha}(t_0), \widehat{\alpha}(t_1))$ and $< \widehat{\alpha}(x), \widehat{\alpha}(y) >\cong N = S_4$.
   Then, $\widehat{\alpha}(G) = \widehat{\alpha}(2^{*4} : S_4) = \widehat{\alpha} < N, t_0 >=< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$ is a homomorphic image of $(2^{*4} : S_4)$.
   Next, we want the homomorphic image of $G = \frac{2^{*4}:S_4}{t_0t_1=(0,1,2)t_0t_2}$.
   Now, $< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >\leq S_{10}$ is a homomorphic image of $G = \frac{2^{*4}:S_4}{t_0t_1=(0,1,2)t_0t_2}$ if the additional relation $t_0t_1 = (0,1,2)t_0t_2$ holds in $< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$.
   Thus, we want to show that $\widehat{\alpha}(t_0t_1) = \widehat{\alpha}((0,1,2)t_0t_2)$
   $\widehat{\alpha}(t_0t_1) = \widehat{\alpha}(t_0)\widehat{\alpha}(t_1) = (1,2)(3,7)(4,8)(5,9)(6,10)(1,3)(4,8)(5,9)(2,6)(7,10)$
   $= (1,6,7)(2,3,10)$.
   $\widehat{\alpha}((0,1,2)t_0t_2) = \widehat{\alpha}((0,1,2))\widehat{\alpha}(t_0)\widehat{\alpha}(t_2)$
   $= (2,3,4)(6,7,8)(1,2)(3,7)(4,8)(5,9)(6,10)(1,4)(5,9)(2,6)(3,7)(8,10)$
   $= (1,6,7)(2,3,10)$.

Therefore, $\widehat{\alpha} : G \to S_{10}$ is a homomorphism.
Now, by First Isomorphism Theorem, we have
$G|ker(\widehat{\alpha}) \cong im(\widehat{\alpha})$
$\Rightarrow \frac{|G|}{|ker(\widehat{\alpha})|} = |im(\widehat{\alpha})| = | < \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) > | = 240$
$\Rightarrow |G| = 240 \times |ker(\widehat{\alpha})|$, where $(|ker(\widehat{\alpha})| \geq 1)$.
$\Rightarrow |G| \geq 240$.
But early we found that $|G| \leq 240$. Therefore, we can conclude that $|G| = 240$. Since $|G| = 240$
$\Rightarrow |ker(\widehat{\alpha})| = 1$.
$\Rightarrow \widehat{\alpha}$ is also an isomorphism.
$\Rightarrow G \cong \widehat{\alpha}(G) =< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$.

It is clear from the Cayley diagram of G over S_4 that every element of G can be written as πw, where π is a permutation of S_4 on four letters, and w is a word in the t_{i_s} of the length at most three. We call this representation of elements of G the symmetric representation. We now give examples of elements of G in both symmetric and permutation representation.

Permutation Representation to Symmetric Presentation

Let $p = (1,6,8)(2,4,10)$ be an element in permutation representation.
We know that $Np = 1^p$.
$\Rightarrow Np = g(1) = 6 = Nt_1t_0$
$\Rightarrow p = nw$, where $n \in N$.
To represent p in the symmetric representation form, we must find $n \in N \cong S_4$.
We have $p(1) = 6$ and Table-3.1 gives that Nt_1t_0 is labeled 6.
If $p = nw$
$\Rightarrow n = pw^{-1}$
$= p(t_1t_0)^{-1}$
$= p\bar{t_0}\bar{t_1}$
$= p\widehat{\alpha}(t_1)\widehat{\alpha}(t_0)$
$= (1,6,8)(2,4,10)(1,3)(4,8)(5,9)(2,6)(7,10)(1,2)(3,7)(4,8)(5,9)(6,10)$
$= (2,4,3)(6,8,7)$.
From our Table-3.1 again,
$n = (0,2,1)$.
Therefore, p can be written in the symmetric representation as follows
$p = (0,2,1)t_0t_1$.

Symmetric Presentation to Permutation Representation

Let $(0,1,2)t_0t_2$ be an element in the symmetric representation.
$\Rightarrow p = \widehat{\alpha}(0,1,2)\widehat{\alpha}(t_0)\widehat{\alpha}(t_2)$
$= (2,3,4)(6,7,8)(1,2)(3,7)(4,8)(5,9)(6,10)(1,4)(5,9)(2,6)(3,7)(8,10)$
$= (1,6,7)(2,3,10)$.
Therefore, $(0,1,2)t_0t_2$ can be written in the permutation representation as
$(0,1,2)t_0t_2 = (1,6,7)(2,3,10)$.

Chapter 4

Construction of A_7

Factoring the progenitor $3^{*5} :_m S_5$ by the relation $[(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_0]^3$. We obtain the finite image

$$G \cong \frac{3^{*5} :_m S_5}{[(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_0]^3}$$

The symmetric presentation of the progenitor $3^{*5} :_m S_5$ can be given by:

$$3^{*5} :_m S_5 \cong < x, y, t | x^5, y^2, (xy)^4, t^3, (yt)^2, (yt^x)^2, (yt^{x^2})^2 >$$

The control group is $N = S_5 \cong < x, y | x^5 = y^2 = (xy)^4 >$. The action on the symmetric generators is given by $x \sim (0,1,2,3,4)$, $y \sim (0,\bar{0})(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})$, and our relations are $[(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_0]^3$ and $[(0,1,2)t_0]^5$.

First, we study our relations.

Relation-1

Let $\pi = (0,\bar{1})(2,\bar{3},4,\bar{2},3,\bar{4})$, then $(\pi t_0)^3 = 1$

$$\begin{aligned}
(\pi t_0)^3 &= 1\\
\pi t_0 \pi \pi \pi^{-1} t_0 \pi t_0 &= 1\\
\pi t_0 \pi^2 (t_0)^{\pi} t_0 &= 1\\
\pi \pi^2 \pi^{-2} t_0 \pi^2 (t_0)^{\pi} t_0 &= 1\\
\pi^3 (t_0)^{\pi^2} (t_0)^{\pi} t_0 &= 1\\
\pi^3 (t_0)^{\pi^2} (t_0)^{\pi} t_0 &= 1
\end{aligned}$$

Since, $\pi = (0,\bar{1})(2,\bar{3},4,\bar{2},3,\bar{4})$
$\Rightarrow \pi^2 = (0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4}) = (2,4,3)(\bar{2},\bar{4},\bar{3})$
$\Rightarrow \pi^3 = (2,4,3)(\bar{2},\bar{4},\bar{3})(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4}) = (0,\bar{1})(\bar{0},1)(2,\bar{2})(3,\bar{3})(4,\bar{4})$.

$$\begin{aligned}
\pi^3(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\Leftrightarrow \pi^3 t_0\bar{t_1}t_0 &= 1\\
\Leftrightarrow \pi^3 t_0\bar{t_1} &= \bar{t_0}\\
\Leftrightarrow \pi^3 t_0 t_1 t_1 &= \bar{t_0}\\
\Leftrightarrow \pi^3 t_0 t_1 &= \bar{t_0}\bar{t_1}
\end{aligned}$$

Relation-2

Let $\pi = (0,1,2)$ then $(\pi t_0)^5 = 1$

$$\begin{aligned}
(\pi t_0)^5 &= 1\\
\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0 &= 1\\
\pi t_0\pi t_0\pi t_0\pi\pi\pi^{-1}t_0\pi t_0 &= 1\\
\pi t_0\pi t_0\pi t_0\pi^2(t_0)^{\pi}t_0 &= 1\\
\pi t_0\pi t_0\pi\pi^2\pi^{-2}t_0\pi^2(t_0)^{\pi}t_0 &= 1\\
\pi t_0\pi t_0\pi^3(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\pi t_0\pi\pi^3\pi^{-3}t_0\pi^3(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\pi t_0\pi^4(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\pi\pi^4\pi^{-4}t_0\pi^4(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\pi^5(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1
\end{aligned}$$

Since $\pi = (0,1,2)$
$\Rightarrow \pi^2 = (0,1,2)(0,1,2) = (0,2,1)$
$\Rightarrow \pi^3 = (0,2,1)(0,1,2) = 1$
$\Rightarrow \pi^4 = \pi^3\pi = (0,1,2)$
$\Rightarrow \pi^5 = \pi^3\pi^2 = \pi^2 = (0,2,1)$

$$\begin{aligned}
\pi^5(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\Leftrightarrow \pi^5 t_1 t_0 t_2 t_1 t_0 &= 1\\
\Leftrightarrow \pi^5 t_1 t_0 t_2 &= \bar{t_0}\bar{t_1}
\end{aligned}$$

Double coset enumeration of G over S_5.

We note that,

$$NeN = \{Nen | n \in N\} = \{Nn | n \in N\} = \{\text{N}\}.$$

Let [*] represents the double coset NeN. [*] has only one distinct right coset, namely N. Since, $N \cong S_5$ acts transitively on $\{0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. Therefore, N has one orbit namely $\{0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. Now, take one representative 0 from the orbit and find which double cosets Nt_0 belongs?

Clearly, $Nt_0 \in Nt_0N = \{Nt_0^n | n \in N\} = \{Nt_0, Nt_1, Nt_2, Nt_3, Nt_4, N\bar{t_0}, N\bar{t_1}, N\bar{t_2}, N\bar{t_3}, N\bar{t_4}\}$, and denote Nt_0N by [0]. Now, consider the coset stabilizer $N^{(0)}$. The coset stabilizer of $N^{(0)}$ is equal to the point stabilizer N^0 and is given by:
$N^{(0)} = N^0 = \{e, (1,4,3)(\bar{1},\bar{4},\bar{3}), (1,3,4)(\bar{1},\bar{3},\bar{4}), (1,2,3)(\bar{1},\bar{2},\bar{3}), (2,4,3)(\bar{2},\bar{4},\bar{3}),$
$(1,4)(2,3)(\bar{1},\bar{4})(\bar{2},\bar{3}), (1,3,2)(\bar{1},\bar{3},\bar{2}), (1,2,4)(\bar{1},\bar{2},\bar{4}), (2,3,4)(\bar{2},\bar{3},\bar{4}), (1,4,2)(\bar{1},\bar{4},\bar{2}),$
$(1,3)(2,4)(\bar{1},\bar{3})(\bar{2},\bar{4}), (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})\}$.
Therefore, the number of single cosets in [0] are

$$\frac{|N|}{|N^{(0)}|} = \frac{5!}{12} = \frac{120}{12} = 10.$$

The orbits of $N^{(0)}$ on $\{0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ are $\{0\}$, $\{1, 2, 3, 4\}$, $\{\bar{0}\}$ and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. Now, take one representative from each orbit, namely 0, 1, $\bar{0}$ and $\bar{1}$, then find the double coset to which Nt_0t_0, Nt_0t_1, $Nt_0\bar{t_0}$ and $Nt_0\bar{t_1}$ belong. However,
$Nt_0t_0 = N\bar{t_0} \in [0]$
$Nt_0t_1 = Nt_0t_1 \in Nt_0t_1N$ (New double coset denote by [01])
$Nt_0\bar{t_0} = N(t_0)^3 = Ne \in [*]$
$Nt_0\bar{t_1} = N\bar{t_0} \in [0]$ (by our Relation-1)
Therefore, one symmetric generator goes back to the identity double coset [*]; five of them stay in [0] and four advance.

Next, we now consider the double coset [01]. By our relation, we have

$$\begin{aligned}
\pi^3 t_0 t_1 &= \bar{t_0}\bar{t_1} \\
(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_0 t_1 &= t_0 t_0 t_1 t_1 \\
\bar{t_0}(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_0 t_1 \bar{t_1} &= \bar{t_0} t_0 t_0 t_1 t_1 \bar{t_1} \\
(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_1 t_0 &= t_0 t_1
\end{aligned}$$

$N(t_1t_0)^{\{(1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3})\}} = Nt_0t_1 \Rightarrow \{(1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3})\} \in N^{(01)}$
$N(t_0t_1)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}} = N\bar{t_0}\bar{t_1} \Rightarrow \{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\} \in N^{(01)}$
$\Rightarrow N^{(01)} =< e, (2,3,4)(\bar{2},\bar{3},\bar{4}) >$
$=< (2,3,4)(\bar{2},\bar{3},\bar{4}), (1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3}), (1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0}) >$
$= \{e, (1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3}), (1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0}), (1,\bar{0})(2,\bar{4},3,\bar{2},4,\bar{3})(0,\bar{1}),$
$(2,4,3)(\bar{2},\bar{4},\bar{3}), (1,0)(2,4)(\bar{1},\bar{0})(\bar{2},\bar{4}), (1,\bar{1})(2,\bar{3})(3,\bar{2})(4,\bar{4})(0,\bar{0}), (1,\bar{0})(2,\bar{3},4,\bar{2},3,\bar{4})(0,\bar{1}),$
$(2,3,4)(\bar{2},\bar{3},\bar{4}), (1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4}), (1,\bar{1})(2,\bar{4})(3,\bar{3})(4,\bar{2})(0,\bar{0}),$
$(1,\bar{0})(2,\bar{2})(3,\bar{3})(4,\bar{4})(1,\bar{1})\}$.
Now, we take (Nt_0t_1) conjugates with the elements of $N^{(01)}$, we obtain the following relation:

$$01 \sim 10 \sim \bar{0}\bar{1} \sim \bar{1}\bar{0}.$$

Therefore, the number of single cosets in the double coset Nt_0t_1N are at most

$$\frac{|N|}{|N^{(01)}|} = \frac{5!}{12} = \frac{120}{12} = 10.$$

Each of single cosets in this double coset has four different names. We now find the four names of ten single cosets in double coset [01].

From our relation, we can identify all the single cosets by conjugating $(01 \sim 10 \sim \bar{0}\bar{1} \sim \bar{1}\bar{0})$ with the transversal elements of $N^{(01)}$ in N. Hence, the ten single cosets of double coset [01] are:

$$\begin{gathered}
01 \sim 10 \sim \bar{0}\bar{1} \sim \bar{1}\bar{0} \\
12 \sim 21 \sim \bar{1}\bar{2} \sim \bar{2}\bar{1} \\
23 \sim 32 \sim \bar{2}\bar{3} \sim \bar{3}\bar{2} \\
34 \sim 43 \sim \bar{3}\bar{4} \sim \bar{4}\bar{3}
\end{gathered}$$

$$24 \sim 42 \sim \bar{2}\bar{4} \sim \bar{4}\bar{2}$$
$$40 \sim 04 \sim \bar{4}\bar{0} \sim \bar{0}\bar{4}$$
$$03 \sim 30 \sim \bar{0}\bar{3} \sim \bar{3}\bar{0}$$
$$14 \sim 41 \sim \bar{1}\bar{4} \sim \bar{4}\bar{1}$$
$$20 \sim 02 \sim \bar{2}\bar{0} \sim \bar{0}\bar{2}$$
$$13 \sim 31 \sim \bar{1}\bar{3} \sim \bar{3}\bar{1}$$

Since the orbits of $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{1,0,\bar{1},\bar{0}\}$ and $\{2,3,4,\bar{2},\bar{3},\bar{4}\}$. We take representative $\bar{1}$ and 2 from each orbits, and determine which double coset $Nt_0t_1t_{\bar{1}}$ and $Nt_0t_1t_2$ belong. However,
$Nt_0t_1t_{\bar{1}} = Nt_0 \in Nt_0N$.
$Nt_0t_1t_2 = N(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_1t_0t_2 = Nt_1t_0t_2 = Nt_{\bar{0}}t_{\bar{1}} \in Nt_0t_1N$.
Therefore, four symmetric generators go back to double coset [0], six stay in the double coset [01] and none of them advance. Hence, we must have completed the manual double coset enumeration of G over S_5. The information is summarized in the Cayley diagram below.

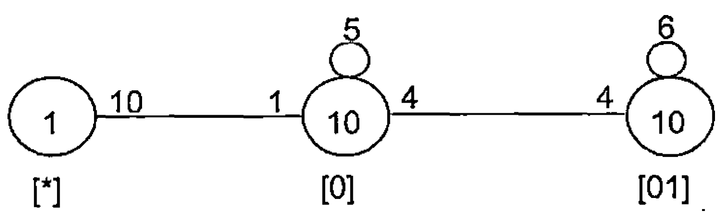

Figure 4.1: The Cayley diagram of A_7 over S_5.

The double coset enumeration shows that the index of $N \cong S_5$ in G is at most
$|G| \leq (1 + 10 + 10)|N| = (21) \times 120 = 2520$.
Now, we are going to show that the $|G|$ is actually equal to 2520.

Permutation Representation.

Cosets	$x \sim (0,1,2,3,4)(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4})$	$y \sim (0,\bar{0})(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})$	t_0
$1.N$	$1.N$	$1.N$	$2.Nt_0$
$2.Nt_0$	$3.Nt_1$	$7.N\bar{t}_0$	$7.N\bar{t}_0$
$3.Nt_1$	$4.Nt_2$	$8.N\bar{t}_1$	$12.N\bar{t}_0\bar{t}_1$
$4.Nt_2$	$5.Nt_3$	$9.N\bar{t}_2$	$13.N\bar{t}_0\bar{t}_2$
$5.Nt_3$	$6.Nt_4$	$11.N\bar{t}_4$	$14.N\bar{t}_0\bar{t}_3$
$6.Nt_4$	$2.Nt_0$	$10.N\bar{t}_3$	$15.N\bar{t}_0\bar{t}_4$
$7.N\bar{t}_0$	$8.N\bar{t}_1$	$2.Nt_0$	$1.N$
$8.N\bar{t}_1$	$9.N\bar{t}_2$	$3.Nt_1$	$3.Nt_1$
$9.N\bar{t}_2$	$10.N\bar{t}_3$	$4.Nt_2$	$4.Nt_2$
$10.N\bar{t}_3$	$11.N\bar{t}_4$	$6.Nt_4$	$5.Nt_3$
$11.N\bar{t}_4$	$7.N\bar{t}_0$	$5.Nt_3$	$6.Nt_4$
$12.Nt_0t_1$	$16.Nt_1t_2$	$12.Nt_0t_1$	$8.N\bar{t}_1$
$13.Nt_0t_2$	$17.Nt_1t_3$	$13.Nt_0t_2$	$9.N\bar{t}_2$
$14.Nt_0t_3$	$18.Nt_1t_4$	$15.Nt_0t_4$	$10.N\bar{t}_3$
$15.Nt_0t_4$	$12.Nt_0t_1$	$14.Nt_0t_3$	$11.N\bar{t}_4$
$16.Nt_1t_2$	$19.Nt_2t_3$	$16.Nt_1t_2$	$16.Nt_1t_2$
$17.Nt_1t_3$	$20.Nt_2t_4$	$18.Nt_1t_4$	$17.Nt_1t_3$
$18.Nt_1t_4$	$13.Nt_0t_2$	$17.Nt_1t_3$	$18.Nt_1t_4$
$19.Nt_2t_3$	$21.Nt_3t_4$	$20.Nt_2t_4$	$19.Nt_2t_3$
$20.Nt_2t_4$	$14.Nt_0t_3$	$19.Nt_2t_3$	$20.Nt_2t_4$
$21.Nt_3t_4$	$15.Nt_0t_4$	$21.Nt_3t_4$	$21.Nt_3t_4$

Table 4.1: Permutation Representation of A_7

Permutation Representation.

Cosets	$(0,\bar{1})(1,\bar{0})(2,\bar{2})(3,\bar{3})(4,\bar{4})$
$1.N$	$1.N$
$2.Nt_0$	$8.N\bar{t_1}$
$3.Nt_1$	$7.N\bar{t_0}$
$4.Nt_2$	$8.N\bar{t_1}$
$5.Nt_3$	$10.N\bar{t_3}$
$6.Nt_4$	$11.N\bar{t_4}$
$7.N\bar{t_0}$	$3.Nt_1$
$8.N\bar{t_1}$	$2.Nt_0$
$9.N\bar{t_2}$	$4.Nt_2$
$10.N\bar{t_3}$	$5.Nt_3$
$11.N\bar{t_4}$	$6.Nt_4$
$12.Nt_0t_1$	$12.Nt_0t_1$
$13.Nt_0t_2$	$16.Nt_1t_2$
$14.Nt_0t_3$	$17.Nt_1t_3$
$15.Nt_0t_4$	$18.Nt_1t_4$
$16.Nt_1t_2$	$13.Nt_0t_2$
$17.Nt_1t_3$	$14.Nt_0t_3$
$18.Nt_1t_4$	$15.Nt_0t_4$
$19.Nt_2t_3$	$19.Nt_2t_3$
$20.Nt_2t_4$	$20.Nt_2t_4$
$21.Nt_3t_4$	$21.Nt_3t_4$

Table 4.2: Permutation Representation of $\{(0,\bar{1})(1,\bar{0})(2,\bar{2})(3,\bar{3})(4,\bar{4})\}$

The homomorphic image of G.

Define $\widehat{\alpha}: G \to S_{21}$, where (21) is a number of single cosets from the Cayley diagram above gives the action of G on the 21 cosets. From the labeling given in Table-4.1, we compute the actions of x, y and t_0 on the 21 cosets.
$\widehat{\alpha}(x) = (2,3,4,5,6)(7,8,9,10,11)(12,16,19,21,15)(13,17,20,14,18)$.
$\widehat{\alpha}(y) = ((2,7)(3,8)(4,9)(5,11)(6,10)(14,15)(17,18)(19,20)$.
$\widehat{\alpha}(t_0) = (1,2,7)(3,12,8)(4,13,9)(5,14,10)(6,15,11)$.
We now verify that $\widehat{\alpha}$ is a homomorphism.

1. We want to show that $\widehat{\alpha}(t_0)$ has exactly 10 conjugates under conjugation by $< \widehat{\alpha}(x), \widehat{\alpha}(y) >$.
   $\widehat{\alpha}(t_0)^{\widehat{\alpha}(x)} = (1,3,8)(4,16,9)(5,17,10)(6,18,11)(2,12,7) = \widehat{\alpha}(t_1)$.
   $\widehat{\alpha}(t_1)^{\widehat{\alpha}(x)} = (1,4,9)(5,19,10)(6,20,11)(2,13,7)(3,16,8) = \widehat{\alpha}(t_2)$.
   $\widehat{\alpha}(t_2)^{\widehat{\alpha}(x)} = (1,5,10)(6,21,11)(2,14,7)(3,17,8)(4,19,9) = \widehat{\alpha}(t_3)$.
   $\widehat{\alpha}(t_3)^{\widehat{\alpha}(x)} = (1,6,11)(2,15,7)(3,18,8)(4,20,9)(5,21,10) = \widehat{\alpha}(t_4)$.
   $\widehat{\alpha}(t_4)^{\widehat{\alpha}(x)} = (1,2,7)(3,12,8)(4,13,9)(5,14,10)(6,15,11) = \widehat{\alpha}(t_0)$.
   $\widehat{\alpha}(\bar{t_0})^{\widehat{\alpha}(x)} = (1,8,3)(4,9,16)(5,10,17)(6,11,18)(2,7,12) = \widehat{\alpha}(\bar{t_1})$.
   $\widehat{\alpha}(\bar{t_1})^{\widehat{\alpha}(x)} = (1,9,4)(5,10,19)(6,11,20)(2,7,13)(3,18,16) = \widehat{\alpha}(\bar{t_2})$.
   $\widehat{\alpha}(\bar{t_2})^{\widehat{\alpha}(x)} = (1,10,5)(6,11,21)(2,7,14)(3,8,17)(4,9,19) = \widehat{\alpha}(\bar{t_3})$.
   $\widehat{\alpha}(\bar{t_3})^{\widehat{\alpha}(x)} = (1,11,6)(2,7,15)(3,8,18)(4,9,20)(5,10,21) = \widehat{\alpha}(\bar{t_4})$.
   $\widehat{\alpha}(\bar{t_4})^{\widehat{\alpha}(x)} = (1,7,2)(3,8,12)(4,9,13)(5,10,14)(6,11,15) = \widehat{\alpha}(\bar{t_0})$.

   $\widehat{\alpha}(t_0)^{\widehat{\alpha}(y)} = (1,7,2)(3,8,12)(4,9,13)(5,10,14)(6,11,15) = \widehat{\alpha}(\bar{t_0})$.
   $\widehat{\alpha}(\bar{t_0})^{\widehat{\alpha}(y)} = (1,2,7)(3,12,8)(4,13,9)(5,14,10)(6,15,11) = \widehat{\alpha}(t_0)$.
   $\widehat{\alpha}(t_1)^{\widehat{\alpha}(y)} = (1,8,3)(4,9,16)(5,10,17)(6,11,18)(2,7,12) = \widehat{\alpha}(\bar{t_1})$.
   $\widehat{\alpha}(\bar{t_1})^{\widehat{\alpha}(y)} = (1,3,8)(4,16,9)(5,17,10)(6,18,11)(2,12,7) = \widehat{\alpha}(t_1)$.
   $\widehat{\alpha}(t_2)^{\widehat{\alpha}(y)} = (1,9,4)(5,10,19)(6,11,20)(2,7,13)(3,18,16) = \widehat{\alpha}(\bar{t_2})$.
   $\widehat{\alpha}(\bar{t_2})^{\widehat{\alpha}(y)} = (1,4,9)(5,19,10)(6,20,11)(2,13,7)(3,16,8) = \widehat{\alpha}(t_2)$.
   $\widehat{\alpha}(t_3)^{\widehat{\alpha}(y)} = (1,11,6)(2,7,15)(3,8,18)(4,9,20)(5,10,21) = \widehat{\alpha}(\bar{t_4})$.
   $\widehat{\alpha}(\bar{t_4})^{\widehat{\alpha}(y)} = (1,5,10)(6,21,11)(2,14,7)(3,17,8)(4,19,9) = \widehat{\alpha}(t_3)$.
   $\widehat{\alpha}(\bar{t_3})^{\widehat{\alpha}(y)} = (1,6,11)(2,15,7)(3,18,8)(4,20,9)(5,21,10) = \widehat{\alpha}(t_4)$.
   $\widehat{\alpha}(t_4)^{\widehat{\alpha}(y)} = (1,10,5)(6,11,21)(2,7,14)(3,8,17)(4,9,19) = \widehat{\alpha}(\bar{t_3})$.

2. $\widehat{\alpha}(N)$ acts as N on $\{\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(t_2), \widehat{\alpha}(t_3), \widehat{\alpha}(t_4), \widehat{\alpha}(\bar{t_0}), \widehat{\alpha}(\bar{t_1}), \widehat{\alpha}(\bar{t_2}), \widehat{\alpha}(\bar{t_3}), \widehat{\alpha}(\bar{t_4})\}$ by conjugation. We note that
$\widehat{\alpha}(x) = \{(\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(t_2), \widehat{\alpha}(t_3), \widehat{\alpha}(t_4))(\widehat{\alpha}(\bar{t_0}), \widehat{\alpha}(\bar{t_1}), \widehat{\alpha}(\bar{t_2}), \widehat{\alpha}(\bar{t_3}), \widehat{\alpha}(\bar{t_4}))\}$,
$\widehat{\alpha}(y) = \{(\widehat{\alpha}(t_0), \widehat{\alpha}(\bar{t_0}))(\widehat{\alpha}(t_1), \widehat{\alpha}(\bar{t_1}))(\widehat{\alpha}(t_2), \widehat{\alpha}(\bar{t_2}))(\widehat{\alpha}(t_3), \widehat{\alpha}(\bar{t_4}))(\widehat{\alpha}(t_4), \widehat{\alpha}(\bar{t_3}))\}$.
and $< \widehat{\alpha}(x), \widehat{\alpha}(y) > \cong N = S_5$.
Then, $\widehat{\alpha}(G) = \widehat{\alpha}(3^{*5} : S_5) = \widehat{\alpha} < N, t_0 > = < \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$ is a homomorphic image of $(3^{*5} : S_5)$.
Next, we want the homomorphic image of $G = \frac{3^{*5}:S_5}{[(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_0]^3}$.
Now, $< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) > \leq S_{21}$ is a homomorphic image of
$G = \frac{3^{*5}:S_5}{[(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_0]^3}$ if the additional relation
$(0,\bar{1})(1,\bar{0})(2,\bar{2})(3,\bar{3})(4,\bar{4})t_0t_1 = \bar{t_0}\bar{t_1}$ holds in $< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$.
Thus, we want to show that $\widehat{\alpha}((0,\bar{1})(1,\bar{0})(2,\bar{2})(3,\bar{3})(4,\bar{4})t_0t_1) = \widehat{\alpha}(\bar{t_0}\bar{t_1})$.
On the left side of our equation
$\widehat{\alpha}((0,\bar{1})(1,\bar{0})(2,\bar{2})(3,\bar{3})(4,\bar{4})t_0t_1) = \widehat{\alpha}((0,\bar{1})(1,\bar{0})(2,\bar{2})(3,\bar{3})(4,\bar{4}))\widehat{\alpha}(t_0)\widehat{\alpha}(t_1)$
$= (2,8)(3,7)(4,9)(5,10)(6,11)(13,16)(14,17)(15,18)(1,2,7)(3,12,8)(4,13,9)$
$(5,14,10)(6,15,11)(1,3,8)(4,16,9)(5,17,10)(6,18,11)(2,12,7)$
$= (1,12)(2,8)(4,16)(5,17)(6,18)(9,13)(10,14)(11,15)$.
On the right side of our equation
$\widehat{\alpha}(\bar{t_0}\bar{t_1}) = \widehat{\alpha}(\bar{t_0})\widehat{\alpha}(\bar{t_1}) = (1,7,2)(3,8,12)(4,9,13)(5,10,14)(6,11,15)(1,8,3)$
$(4,9,16)(5,10,17)(6,11,18)(2,7,12)$
$= (1,12)(2,8)(4,16)(5,17)(6,18)(9,13)(10,14)(11,15)$

Therefore, $\widehat{\alpha} : G \to S_{21}$ is a homomorphism map. Now by First Isomorphism Theorem, we have
$G|ker(\widehat{\alpha}) \cong im(\widehat{\alpha})$
$\Rightarrow \frac{|G|}{|ker(\widehat{\alpha})|} = |im(\widehat{\alpha})| = | < \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0)| = 2520$
$\Rightarrow |G| = 2520 \times |ker(\widehat{\alpha})|$, where $(|ker(\widehat{\alpha})| \geq 1)$.
$\Rightarrow |G| \geq 2520$.
But early we found that $|G| \leq 2520$. Therefore, we can conclude that $|G| = 2520$. Since $|G| = 2520$,
$\Rightarrow |ker(\widehat{\alpha})| = 1$.
$\Rightarrow \widehat{\alpha}$ is also an isomorphism.

$\Rightarrow G \cong \widehat{\alpha}(G) =< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$.

It is clear from the Cayley diagram of G over S_5 that every element of G can be written as πw, where π is a permutation of S_5 on ten letters, and w is a word in the t_is of the length at most two. We call this representation of elements of G the symmetric representation. We now give examples of elements of G in both symmetric and permutation representation.

Permutation Representation to Symmetric Presentation

Let $p = (1,19,7,16,15,3,21)(2,17,11,4,14,8,20)(5,10,9,13,12,18,6)$ be an element in permutation representation.
We know that $Np = Nw$.
$\Rightarrow Np = p(1) = 19 = Nt_2t_3$
$\Rightarrow p = nw$, where $n \in N$.
In order the represent p in the symmetric representation form, we must find $n \in N \cong S_5$.
We have $p(1) = 19$ and Table-4.1 gives that Nt_2t_3 labeled 19.
$\Rightarrow Np = Nt_2t_3$.
$p = nw$
$n = pw^{-1}$
$= p(t_2t_3)^{-1}$
$= p\bar{t_3}\bar{t_2}$
$= p\widehat{\alpha}(\bar{t_3})\widehat{\alpha}(\bar{t_2}) = (1,19,7,16,15,3,21)(2,17,11,4,14,8,20)(5,10,9,13,12,18,6)$
$(1,5,10)(2,14,7)(3,17,8)(4,19,9)(6,21,11)(1,4,9)(2,13,7)(3,16,8)(5,19,10)(6,20,11)$
$= (2,3,6,5,4)(7,8,11,10,9)(12,18,21,19,13)(14,16,15,17,20)$.
From our Table-4.1 again,
$n = (0,1,4,3,2)(\bar{0},\bar{1},\bar{4},\bar{3},\bar{2})$.
Therefore, p can be written in the symmetric representation as follows
$p = (0,1,4,3,2)(\bar{0},\bar{1},\bar{4},\bar{3},\bar{2})t_2t_3$.

Symmetric Presentation to Permutation Representation

Let $\{(0,\bar{1})(1,\bar{0})(2,\bar{2})(3,\bar{3})(4,\bar{4})t_0t_1\}$ be an element in the symmetric representation.
$\Rightarrow p = \widehat{\alpha}(0,\bar{1})(1,\bar{0})(2,\bar{2})(3,\bar{3})(4,\bar{4})\widehat{\alpha}(t_0)\widehat{\alpha}(t_1)$
$= (2,8)(3,7)(4,9)(5,10)(6,11)(13,16)(14,17)(15,18)(1,2,7)(3,12,8)(4,13,9)$

$(5,14,10)(6,15,11)(1,3,8)(4,16,9)(5,17,10)(6,18,11)(2,12,7)$
$= (1,12)(2,8)(4,16)(5,17)(6,18)(9,13)(10,14)(11,15)$.

Therefore, $\{(0,\bar{1})(1,\bar{0})(2,\bar{2})(3,\bar{3})(4,\bar{4})t_0t_1\}$ can be written in the permutation representation as

$(0,\bar{1})(1,\bar{0})(2,\bar{2})(3,\bar{3})(4,\bar{4})t_0t_1 = (1,12)(2,8)(4,16)(5,17)(6,18)(9,13)(10,14)(11,15)$.

Chapter 5

Construction of $A_7 \times S_3$

5.1 Manual double coset enumeration of G over S_5

Factoring the progenitor $3^{*5} :_m S_5$ by the relation $[(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_0]^4$. We obtain the finite image

$$G \cong \frac{3^{*5} :_m S_5}{[(0,\bar{1})(0,1)(2,\bar{3},4,2,3,4)t_2]^4}$$

The symmetric presentation of the progenitor $3^{*5} :_m S_5$ can be given by:

$$3^{*5} :_m S_5 \cong < x, y, t | x^5, y^2, (xy)^4, t^3, (yt)^2, (yt^x)^2, (yt^{x^2})^2 >$$

The control group is $N = S_5 \cong < x, y | x^5 = y^2 = (xy)^4 >$. The action on the symmetric generators is given by $x \sim (0,1,2,3,4)(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4})$, $y \sim (0,\bar{0})(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})$, and our relation is $[(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_0]^4$.

First, we study our relation.

Let $\pi = (0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})$, then $(\pi t_0)^4 = 1$

$$\begin{aligned}
(\pi t_2)^4 &= 1\\
\pi t_2\pi t_2\pi t_2\pi t_2 &= 1\\
\pi t_2\pi t_2\pi\pi\pi^{-1}t_2\pi t_2 &= 1\\
\pi t_2\pi t_2\pi^2(t_2)^{\pi}t_2 &= 1\\
\pi t_2\pi\pi^2\pi^{-2}t_2\pi^2(t_2)^{\pi}t_2 &= 1\\
\pi\pi^3\pi^{-3}t_2\pi^3(t_2)^{\pi^2}(t_2)^{\pi}t_2 &= 1\\
\pi^4(t_2)^{\pi^3}(t_2)^{\pi^2}(t_2)^{\pi}t_2 &= 1
\end{aligned}$$

Since, $\pi = (0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})$

$\Rightarrow \pi^2 = (0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4}) = (2,4,3)(\bar{2},\bar{4},\bar{3})$.

$\Rightarrow \pi^3 = (2,4,3)(\bar{2},\bar{4},\bar{3})(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4}) = (0,\bar{1})(\bar{0},1)(2,\bar{2})(3,\bar{3})(4,\bar{4})$.

$\Rightarrow \pi^4 = (0,\bar{1})(\bar{0},1)(2,\bar{2})(3,\bar{3})(4,\bar{4})(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4}) = (2,3,4)(\bar{2},\bar{3},\bar{4})$.

$$\begin{aligned}
\pi^4(t_2)^{\pi^3}(t_2)^{\pi^2}(t_2)^{\pi}t_2 &= 1\\
\Leftrightarrow \pi^4\bar{t_2}t_4\bar{t_3}t_2 &= 1\\
\Leftrightarrow \pi^4\bar{t_2}t_4 &= \bar{t_2}t_3
\end{aligned}$$

Double coset enumeration of G over S_5.

Unless otherwise specified, all the lemmas using in Chapter-5 can be found from page (51-54).

We note that,

$$NeN = \{Nen|n \in N\} = \{Nn|n \in N\} = \{\mathrm{N}\}.$$

Let [*] represents the double coset NeN. [*] has only one distinct single coset, namely N. Since, $N \cong S_5$, it is transitive on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$. Therefore, N has one orbit namely $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ in $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$. Now, we take a representative $\bar{2}$ from the orbit and find which double coset $N\bar{t_2}$ belongs.

Clearly, $N\bar{t_2} \in N\bar{t_2}N = \{N\bar{t_2}^n | n \in N\} = \{Nt_0, Nt_1, Nt_2, Nt_3, Nt_4, N\bar{t_0}, N\bar{t_1}, N\bar{t_2}, N\bar{t_3}, N\bar{t_4}\}$, and denote $N\bar{t_2}N$ by $[\bar{2}]$. Now, consider the coset stabilizer $N^{(\bar{2})}$. The coset stabilizer $N^{(\bar{2})}$ is equal to the point stabilizer $N^{\bar{2}}$ and is given by:

$$N^{(\bar{2})} = N^{\bar{2}} = \{Nt_0, Nt_1, Nt_2, Nt_3, Nt_4, N\bar{t_0}, N\bar{t_1}, N\bar{t_2}, N\bar{t_3}, N\bar{t_4}\}.$$

Therefore, the number of single cosets in $[\bar{2}]$ are

$\frac{|N|}{|N^{(\bar{2})}|} = \frac{5!}{12} = \frac{120}{12} = 10$. Each single coset has only one name.

The orbits of $N^{(\bar{2})}$ on $\{0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ are $\{2\}$, $\{1, 3, 4, 5\}$, $\{\bar{2}\}$ and $\{\bar{1}, \bar{3}, \bar{4}, \bar{5}\}$. Now, take one representative from each orbit, namely 2, 4, $\bar{2}$ and $\bar{4}$, then find the double coset to which $N\bar{t_2}t_2$, $N\bar{t_2}t_4$, $N\bar{t_2}\bar{t_2}$ and $N\bar{t_2}\bar{t_4}$ belong. However,

$N\bar{t_2}t_2 = Ne \in [*]$

$N\bar{t_2}t_4 = N\bar{t_2}t_4 \in N\bar{t_2}t_4N$ (New double coset denote by $[\bar{2}4]$)

$N\bar{t_2}\bar{t_2} = Nt_2 \in [\bar{2}]$

$N\bar{t_2}\bar{t_4} = N\bar{t_2}\bar{t_4} \in N\bar{t_2}\bar{t_4}N$ (New double coset denote by $[\bar{2}\bar{4}]$)

Therefore, one symmetric generator goes back to the identity double coset $[*]$; one stays in $[\bar{2}]$; four extend to new double coset $[\bar{2}4]$, and the other four extend to new double coset $[\bar{2}\bar{4}]$.

Next, we now consider the double coset $[\bar{2}4]$. By our relation, we have

$N(\bar{t_2}t_4)^{\{(1,4,3)(\bar{1},\bar{4},\bar{3})\}} = N\bar{t_2}t_3 \Rightarrow (1,4,3)(\bar{1},\bar{4},\bar{3}) \in N^{(\bar{2}4)}$

Since, $N^{(\bar{2}4)} \geq N^{\bar{2}4} =< e, (1,3,0)(\bar{1},\bar{3},\bar{0}) >$

$\Rightarrow N^{(\bar{2}4)} =< (1,3,0)(\bar{1},\bar{3},\bar{0}), (1,4,3)(\bar{1},\bar{4},\bar{3}) >$

$= \{e, (1,3,0)(\bar{1},\bar{3},\bar{0}), (1,4,3)(\bar{1},\bar{4},\bar{3}), (1,0,3)(\bar{1},\bar{0},\bar{3}), (3,0,4)(\bar{3},\bar{0},\bar{4}),$
$(1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0}), (1,4)(3,0)(\bar{1},\bar{4})(\bar{3},\bar{0}), (1,0,4)(\bar{1},\bar{0},\bar{4}), (3,4,0)(\bar{3},\bar{4},\bar{0}),$
$(1,3,4)(\bar{1},\bar{3},\bar{4}), (1,4,0)(\bar{1},\bar{4},\bar{0}), (1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4})\}$.

We now take $N\bar{t_2}t_4$ conjugates with the elements of $N^{(\bar{2}4)}$, we obtaine the following relations:

$$\bar{2}4 \sim \bar{2}3 \sim \bar{2}1 \sim \bar{2}0.$$

Therefore, the number of single cosets in the double coset $(N\bar{t_2}t_4N)$ are at most

$$\frac{|N|}{|N^{(24)}|}=\frac{5!}{12}=\frac{120}{12}=10.$$ Each single coset has four different names.

We can identify the ten single cosets in $\bar{2}4)$ by conjugating ($\bar{2}4\sim\bar{2}3\sim\bar{2}1\sim\bar{2}0$) with the elements in the transversal of $N^{(\bar{2}4)}$.
Hence, the ten distinct single single cosets are:

$$\begin{aligned}
&\bar{2}4\sim\bar{2}3\sim\bar{2}1\sim\bar{2}0\\
&\bar{3}0\sim\bar{3}4\sim\bar{3}2\sim\bar{3}1\\
&2\bar{3}\sim2\bar{4}\sim2\bar{1}\sim2\bar{0}\\
&\bar{4}1\sim\bar{4}0\sim\bar{4}3\sim\bar{4}2\\
&4\bar{0}\sim4\bar{3}\sim4\bar{2}\sim4\bar{1}\\
&3\bar{4}\sim3\bar{0}\sim3\bar{2}\sim3\bar{1}\\
&\bar{0}1\sim\bar{0}4\sim\bar{0}2\sim\bar{0}3\\
&0\bar{1}\sim0\bar{4}\sim0\bar{3}\sim0\bar{2}\\
&\bar{1}3\sim\bar{1}2\sim\bar{1}0\sim\bar{1}4\\
&1\bar{2}\sim1\bar{3}\sim1\bar{0}\sim1\bar{4}
\end{aligned}$$

Since the orbits of $N^{(\bar{2}4)}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{2\}$, $\{\bar{2}\}$, $\{1,3,4,0\}$, and $\{\bar{1},\bar{3},\bar{4},\bar{0}\}$. We now take representatives 2, $\bar{2}$, 4, and $\bar{4}$ from the orbits, and determine which double coset $N\bar{t_2}t_4t_2$, $N\bar{t_2}t_4\bar{t_2}$, $N\bar{t_2}t_4t_4$ and $N\bar{t_2}t_4\bar{t_4}$ belong. However,
$N\bar{t_2}t_4t_2 \in N\bar{t_2}t_4t_2N$ (New double coset denote by $[\bar{2}42]$).
$N\bar{t_2}t_4\bar{t_2} \in N\bar{t_2}t_4\bar{t_2}N$ (New double coset denote by $[\bar{2}4\bar{2}]$).
$N\bar{t_2}t_4t_4 = N\bar{t_2}\bar{t_4} \in [\bar{2}\bar{4}]$.
$N\bar{t_2}t_4\bar{t_4} = N\bar{t_2} \in [\bar{2}]$.
Therefore, four symmetric generators go back to the double coset $[\bar{2}]$; four extend to $[\bar{2}\bar{4}]$; one extends to new double coset $[\bar{2}42]$, and another one extends to new double coset $[\bar{2}4\bar{2}]$.

Now, consider the double coset $[\bar{2}\bar{4}]$. Since the coset stabilizer is equal to its point stabilizer, we have
$N^{(\bar{2}\bar{4})} = N^{\bar{2}\bar{4}} =<(1,3,0)(\bar{1},\bar{3},\bar{0})>=\{e,(1,3,0)(\bar{1},\bar{3},\bar{0}),(1,0,3)(\bar{1},\bar{0},\bar{3})\}$.
Therefore, the number of single cosets in the double coset $N\bar{t_2}\bar{t_4}N$ are at most

$$\frac{|N|}{|N^{(\bar{2}\bar{4})}|}=\frac{5!}{3}=\frac{120}{3}=40.$$ Each single coset has only one name.

We can identify all the forty single cosets in $[\bar{2}\bar{4}]$ by conjugating $N\bar{t_2}\bar{t_4}$ with the transversal of $N^{(\bar{2}\bar{4})}$. Hence, the forty single cosets of double coset $[\bar{2}\bar{4}]$ are:
$\bar{2}\bar{4}, \bar{3}\bar{0}, 23, \bar{4}\bar{1}, 40, 34, \bar{0}\bar{2}, 31, 01, \bar{4}\bar{3}, \bar{1}\bar{3}, 02, 42, 12, \bar{0}\bar{1}, \bar{0}\bar{4}, 14, 13, 03, \bar{3}\bar{2}, \bar{1}\bar{2}, \bar{1}\bar{0}, 20, 24, \bar{1}\bar{4}, \bar{2}\bar{3},$
$\bar{2}\bar{1}, 10, \bar{2}\bar{0}, 30, \bar{3}\bar{4}, 21, \bar{3}\bar{1}, 41, \bar{4}\bar{0}, 43, 32, \bar{4}\bar{2}, 04, \bar{0}\bar{3}.$

The orbits of $N^{(\bar{2}\bar{4})}$ on $\{0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ are $\{2\}$, $\{\bar{2}\}$, $\{4\}$, $\{\bar{4}\}$, $\{1, 3, 0\}$ and $\{\bar{1}, \bar{3}, \bar{0}\}$. Now, we take representatives 2, $\bar{2}$, 4, $\bar{4}$, 3, and $\bar{3}$ from each orbit which, and determine which double coset $N\bar{t_2}\bar{t_4}t_2$, $N\bar{t_2}\bar{t_4}\bar{t_2}$, $N\bar{t_2}\bar{t_4}t_4$, $N\bar{t_2}\bar{t_4}\bar{t_4}$, $N\bar{t_2}\bar{t_4}t_3$ and $N\bar{t_2}\bar{t_4}\bar{t_3}$ belong. However,
$N\bar{t_2}\bar{t_4}t_2 \in N\bar{t_2}\bar{t_4}t_2N$ (New double coset denote by $[\bar{2}\bar{4}2]$)
$N\bar{t_2}\bar{t_4}\bar{t_2} \in N\bar{t_2}\bar{t_4}\bar{t_2}N$ (New double coset denote by $[\bar{2}\bar{4}\bar{2}]$)
$N\bar{t_2}\bar{t_4}t_4 = N\bar{t_2} \in [\bar{2}]$
$N\bar{t_2}\bar{t_4}\bar{t_4} = N\bar{t_2}t_4 \in [\bar{2}4]$
$N\bar{t_2}\bar{t_4}t_3 \in [\bar{2}\bar{4}]$ **(By Lemma-I)**
$N\bar{t_2}\bar{t_4}\bar{t_3} \in N\bar{t_2}\bar{t_4}\bar{t_3}N$ (New double coset denote by $[\bar{2}\bar{4}\bar{3}]$)

Therefore, one symmetric generator extends to new double coset $[\bar{2}\bar{4}2]$); one extends to $[\bar{2}\bar{4}\bar{2}]$); three extend to $[\bar{2}\bar{4}\bar{3}]$). For the other five symmetric generators, one goes back to $[\bar{2}]$; one goes to $[\bar{2}4]$; and three stays in $[\bar{2}\bar{4}]$.

Next, consider the double coset $[\bar{2}\bar{4}2]$.
Claim: $\bar{2}\bar{4}2 \sim \bar{1}\bar{4}1$

Proof. By our relation $(2, 3, 4)(\bar{2}, \bar{3}, \bar{4})\bar{2}4 = \bar{2}3$ conjugating with an element $(1, 0, 3, 2, 4)(\bar{1}, \bar{0}, \bar{3}, \bar{2}, \bar{4}) \in N$, we obtain the following relation $(4, 2, 1)(\bar{4}, \bar{2}, \bar{1})\bar{4}1 = \bar{4}2$.
$\Rightarrow \bar{2}\underline{\bar{4}2} = \bar{2}(4, 2, 1)(\bar{4}, \bar{2}, \bar{1})\bar{4}1 = (4, 2, 1)(\bar{4}, \bar{2}, \bar{1})\bar{1}\bar{4}1$ $\square$

Since, $N^{(\bar{2}\bar{4}2)} \geq N^{\bar{2}\bar{4}2} = \{e, (1, 3, 0)(\bar{1}, \bar{3}, \bar{0})\}$, and
$N(\bar{t_2}\bar{t_4}t_2)^{\{(1,0,2)(\bar{1},\bar{0},\bar{2})\}} = N\bar{t_1}\bar{t_4}t_1 \Rightarrow \{(1, 0, 2)(\bar{1}, \bar{0}, \bar{2})\} \in N^{(\bar{2}\bar{4}2)}$
$\Rightarrow N^{(\bar{2}\bar{4}2)} = < (1, 3, 0)(\bar{1}, \bar{3}, \bar{0}), (1, 0, 2)(\bar{1}, \bar{0}, \bar{2}) >$
$= \{e, (1, 3, 0)(\bar{1}, \bar{3}, \bar{0}), (1, 0, 2)(\bar{1}, \bar{0}, \bar{2}), (1, 2, 0)(\bar{1}, \bar{2}, \bar{0}), (2, 0, 3)(\bar{2}, \bar{0}, \bar{3}),$
$(1, 3, 2)(\bar{1}, \bar{3}, \bar{2}), (1, 0, 3)(\bar{1}, \bar{0}, \bar{3}), (1, 2)(3, 0)(\bar{1}, \bar{2})(\bar{3}, \bar{0}), (2, 3, 0)(\bar{2}, \bar{3}, \bar{0}),$
$(1, 3)(2, 0)(\bar{1}, \bar{3})(\bar{2}, \bar{0}), (1, 0)(2, 3)(\bar{1}, \bar{0})(\bar{2}, \bar{3}), (1, 2, 3)(\bar{1}, \bar{2}, \bar{3})\}.$
Now, take $N\bar{t_2}\bar{t_4}t_2$ conjugates with the elements of $N^{(\bar{2}\bar{4}2)}$ we obtain the following relation:

$$\bar{2}\bar{4}2 \sim \bar{1}\bar{4}1 \sim \bar{0}\bar{4}0 \sim \bar{3}\bar{4}3$$

Therefore, the number of single cosets in the double coset $N\bar{t_2}\bar{t_4}t_2N$ are at most

$\frac{|N|}{|N^{(\bar{2}\bar{4}2)}|} = \frac{5!}{12} = \frac{120}{12} = 10$. Each single coset has four different names.

Again, from the relation above, we can identity all the ten single cosets by conjugating ($\bar{2}\bar{4}2 \sim \bar{1}\bar{4}1 \sim \bar{0}\bar{4}0 \sim \bar{3}\bar{4}3$) with elements in the transversal of $N^{(\bar{2}\bar{4}2)}$. Hence, the ten distinct right single cosets are:

$$\begin{gathered}
\bar{2}\bar{4}2 \sim \bar{1}\bar{4}1 \sim \bar{0}\bar{4}0 \sim \bar{3}\bar{4}3 \\
\bar{3}03 \sim \bar{2}02 \sim \bar{1}01 \sim \bar{4}04 \\
23\bar{2} \sim 13\bar{1} \sim 03\bar{0} \sim 43\bar{4} \\
\bar{4}\bar{1}4 \sim \bar{3}\bar{1}3 \sim \bar{2}\bar{1}2 \sim \bar{0}\bar{1}0 \\
40\bar{4} \sim 20\bar{2} \sim 10\bar{1} \sim 30\bar{3} \\
34\bar{3} \sim 24\bar{2} \sim 14\bar{1} \sim 04\bar{0} \\
\bar{3}\bar{2}3 \sim \bar{1}\bar{2}1 \sim \bar{0}\bar{2}0 \sim \bar{4}\bar{2}4 \\
31\bar{3} \sim 21\bar{2} \sim 01\bar{0} \sim 41\bar{4} \\
\bar{1}\bar{3}1 \sim \bar{0}\bar{3}0 \sim \bar{4}\bar{3}4 \sim \bar{2}\bar{3}2 \\
12\bar{1} \sim 02\bar{0} \sim 42\bar{4} \sim 32\bar{3}
\end{gathered}$$

The orbits of $N^{(\bar{2}\bar{4}2)}$ over $\{0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ are $\{4\}$, $\{\bar{4}\}$, $\{1, 2, 3, 0\}$ and $\{\bar{1}, \bar{2}, \bar{3}, \bar{0}\}$. We now take representatives 4, $\bar{4}$, 2 and $\bar{2}$ from each orbit, and determine which double coset $N\bar{t_2}\bar{t_4}t_2t_4$, $N\bar{t_2}\bar{t_4}t_2\bar{t_4}$, $N\bar{t_2}\bar{t_4}t_2t_2$ and $N\bar{t_2}\bar{t_4}t_2\bar{t_2}$ belong. However,
$N\bar{t_2}\bar{t_4}t_2t_4 \in N\bar{t_2}\bar{t_4}t_2t_4N$ (New double coset denote by $[\bar{2}\bar{4}24]$)
$N\bar{t_2}\bar{t_4}t_2\bar{t_4} \in [\bar{2}\bar{4}2]$ **(By Lemma-II)**
$N\bar{t_2}\bar{t_4}t_2t_2 = N\bar{t_2}\bar{t_4}\bar{t_2} \in [\bar{2}\bar{4}\bar{2}]$
$N\bar{t_2}\bar{t_4}t_2\bar{t_2} = N\bar{t_2}\bar{t_4} \in [\bar{2}\bar{4}]$

Now, consider the new double coset $[\bar{2}\bar{4}24]$.
Claim: $\bar{2}\bar{4}24 \sim \bar{1}\bar{4}14 \sim 010\bar{1}$

Proof. By our relation $(2, 3, 4)(\bar{2}, \bar{3}, \bar{4})\bar{2}4 = \bar{2}3$ conjugating with an element of N, we obtained the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,0,3,2,4)(\bar{1},\bar{0},\bar{3},\bar{2},\bar{4})\}} = (\bar{2}3)^{\{(1,0,3,2,4)(\bar{1},\bar{0},\bar{3},\bar{2},\bar{4})\}}$
  $\Rightarrow (4,2,1)(\bar{4},\bar{2},\bar{1})\bar{4}1 = \bar{4}2$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}} = (\bar{2}3)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}}$
  $\Rightarrow (\bar{2},\bar{4},\bar{3})(2,4,3)2\bar{3} = 2\bar{4}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{2},3,\bar{4})(2,\bar{3},4,\bar{1})(0,\bar{0})\}} = (\bar{2}3)^{\{(1,\bar{2},3,\bar{4})(2,\bar{3},4,\bar{1})(0,\bar{0})\}}$
  $\Rightarrow (\bar{3},\bar{4}bar1)(3,4,1)3\bar{1} = 3\bar{4}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(2,4,0)(\bar{2},\bar{4},\bar{0})\}} = (\bar{2}3)^{\{(2,4,0)(\bar{2},\bar{4},\bar{0})\}}$
  $\Rightarrow (4,3,0)(\bar{4},\bar{3},\bar{0})\bar{2}0 = \bar{2}3$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{0},3,\bar{2})(2,\bar{1},0,\bar{3})(4,\bar{4})\}} = (\bar{2}3)^{\{(1,\bar{0},3,\bar{2})(2,\bar{1},0,\bar{3})(4,\bar{4})\}}$
  $\Rightarrow (\bar{1},\bar{2},\bar{4})(1,2,4)1\bar{4} = 1\bar{2}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,0,3,4,2)(\bar{1},\bar{0},\bar{3},\bar{4},\bar{2})\}} = (\bar{2}3)\{(1,0,3,4,2)(\bar{1},\bar{0},\bar{3},\bar{4},\bar{2})\}$
  $\Rightarrow (1,4,2)(\bar{1},\bar{4},\bar{2})\bar{1}2 = \bar{1}4$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0})\}} = (\bar{2}3)^{\{(1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0})\}}$
  $\Rightarrow (1,0,4)(\bar{1},\bar{0},\bar{4})\bar{1}4 = \bar{1}0$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{2},0,\bar{3})(1,\bar{0},3,\bar{1})(4,\bar{4})\}} = (\bar{2}3)^{\{(1,\bar{2},0,\bar{3})(1,\bar{0},3,\bar{1})(4,\bar{4})\}}$
  $\Rightarrow (\bar{0},\bar{1},\bar{4})(0,1,4)0\bar{4} = 0\bar{1}$

From the relations above, we have

1. $\bar{2}\underline{\bar{4}2}4 = \bar{2}(4,2,1)(\bar{4},\bar{2},\bar{1})\bar{4}14 = (4,2,1)(\bar{4},\bar{2},\bar{1})\bar{1}\bar{4}14.$

2. $\bar{2}\bar{4}24 = \bar{2}\bar{4}\underline{2\bar{4}}\bar{4} = \bar{2}\bar{4}(\bar{2},\bar{4},\bar{3})(2,4,3)2\bar{3}\bar{4} = (\bar{2},\bar{4},\bar{3})(2,4,3)\bar{4}\bar{3}2\bar{3}\bar{4}$
$= (\bar{2},\bar{4},\bar{3})(2,4,3)\bar{4}\bar{3}23\underline{3\bar{4}} = (\bar{2},\bar{4},\bar{3})(2,4,3)\bar{4}\bar{3}23(\bar{3},\bar{4},\bar{1})(3,4,1)3\bar{1}$
$= (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})\bar{1}\bar{4}2\bar{4}\underline{\bar{4}3}\bar{1} = (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})\bar{1}\bar{4}2\bar{4}(4,3,0)(\bar{4},\bar{3},\bar{0})\bar{4}0\bar{1}$
$= (1,2)(4,0)(\bar{1},\bar{2})(\bar{4},\bar{0})\bar{1}3\underline{2\bar{3}}\bar{4}0\bar{1} = (1,2)(4,0)(\bar{1},\bar{2})(\bar{4},\bar{0})\bar{1}\bar{1}(\bar{2}\bar{3}\bar{4})(2,3,4)2\bar{4}\bar{4}0\bar{1}$
$= (1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0})\bar{1}\bar{4}240\bar{1} = (1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0})1\underline{1\bar{4}}240\bar{1}$
$= (1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0})1(\bar{1},\bar{4},\bar{2})(1,4,2)1\bar{2}240\bar{1} = (1,3,4)(\bar{1},\bar{3},\bar{4})4140\bar{1}$
$= (1,3,4)(\bar{1},\bar{3},\bar{4})4\bar{1}\underline{\bar{1}4}0\bar{1} = (1,3,4)(\bar{1},\bar{3},\bar{4})4\bar{1}(1,4,0)(\bar{1},\bar{4},\bar{0})\bar{4}00\bar{1}$
$= (0,3,4)(\bar{0},\bar{3},\bar{4})\underline{0\bar{4}}\bar{1}0\bar{1} = (0,3,4)(\bar{0},\bar{3},\bar{4})(0,4,1)(\bar{0}\bar{4}\bar{1})0\bar{1}\bar{1}\bar{0}\bar{1}$
$= (1,3,4)(\bar{1},\bar{3},\bar{4})01\bar{0}\bar{1}).$ □

Since, $N^{(\bar{2}\bar{4}24)} \geq N^{\bar{2}\bar{4}24} =< (1,3,0)(\bar{1},\bar{3},\bar{0}) >$ and
$N(\bar{t_2}\bar{t_4}t_2t_4)^{\{(1,\bar{3},4,\bar{1},3,\bar{4})(2,\bar{0})(0,\bar{2})\}} = Nt_0t1\bar{t_0}\bar{t_1} \Rightarrow (1,\bar{3},4,\bar{1},3,\bar{4})(2,\bar{0})(0,\bar{2}) \in N^{(\bar{2}\bar{4}24)}$
$\Rightarrow N^{(\bar{2}\bar{4}24)} =< (1,3,0)(\bar{1},\bar{3},\bar{0}), (1,\bar{3},4,\bar{1},3,\bar{4})(2,\bar{0})(0,\bar{2}) >\cong S_5$.

We now take $N\bar{t_2}\bar{t_4}t_2t_4$ conjugates with the elements of $N^{(\bar{2}\bar{4}24)}$, we obtain the following relation:
$14\bar{4}\bar{1} \sim 04\bar{0}\bar{4} \sim \bar{3}\bar{1}31 \sim 1\bar{1}\bar{1}01 \sim 42\bar{4}\bar{2} \sim \bar{0}\bar{4}04 \sim \bar{0}\bar{3}03 \sim \bar{2}\bar{0}20 \sim 23\bar{2}\bar{3} \sim 43\bar{4}\bar{3} \sim 32\bar{3}\bar{2} \sim 10\bar{1}\bar{0} \sim \bar{1}\bar{0}10 \sim 24\bar{2}\bar{4} \sim 31\bar{3}\bar{1} \sim \bar{2}\bar{4}24 \sim \bar{4}\bar{0}40 \sim 30\bar{3}\bar{0} \sim 34\bar{3}\bar{4} \sim \bar{1}\bar{2}12 \sim \bar{4}\bar{3}43 \sim \bar{4}\bar{1}41 \sim \bar{1}\bar{3}13 \sim 13\bar{1}\bar{3} \sim 40\bar{4}\bar{0} \sim 21\bar{2}\bar{1} \sim 20\bar{2}\bar{0} \sim \bar{3}\bar{2}32 \sim 02\bar{0}\bar{2} \sim \bar{3}\bar{0}30 \sim 12\bar{1}\bar{2} \sim \bar{1}\bar{4}14 \sim \bar{0}\bar{2}02 \sim 41\bar{4}\bar{1} \sim 01\bar{0}\bar{1} \sim \bar{2}\bar{1}21 \sim \bar{3}\bar{4}34 \sim \bar{4}\bar{2}42 \sim 03\bar{0}\bar{3} \sim \bar{2}\bar{3}23$.

Therefore, the number of single cosets in the double coset $N\bar{t_2}\bar{t_4}t_2N$ are at most

$\frac{|N|}{|N^{(\bar{2}\bar{4}24)}|} = \frac{5!}{120} = \frac{120}{120} = 1$. This single coset has forty different names.

Since, $N^{(\bar{2}\bar{4}24)}$ acts transitively over $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$, therefore, all ten symmetric generators go back to the double coset $[\bar{2}\bar{4}2]$.

Next, consider the the double coset $[\bar{2}\bar{4}\bar{2}]$.

Claim: $\bar{2}\bar{4}\bar{2} \sim \bar{0}\bar{4}\bar{0}$

Proof. By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$ conjugating with an elements of N, we obtain the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{1})(2,\bar{4},0,\bar{3})(3,\bar{2},4,\bar{0})\}} = (\bar{2}3)^{\{(1,\bar{1})(2,\bar{4},0,\bar{3})(3,\bar{2},4,\bar{0})\}}$
  $\Rightarrow (4,2,0)(\bar{4},\bar{2},\bar{0})4\bar{0} = 4\bar{2}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,3,4,2,0)(\bar{1},\bar{3},\bar{4}\,bar2,\bar{0})\}} = (\bar{2}3)^{\{(1,3,4,2,0)(\bar{1}\bar{3}\bar{4}\bar{2}\bar{0})\}}$
  $\Rightarrow (0,2,4)(\bar{0},\bar{2},\bar{4})\bar{0}4 = \bar{0}2$

$\Rightarrow \bar{2}\bar{4}\bar{2} = \bar{2}4\underline{4\bar{2}} = \bar{2}4(4,2,0)(\bar{4},\bar{2},\bar{0})4\bar{0} = \underline{\bar{0}2}4\bar{0} = (0,2,4)(\bar{0},\bar{2},\bar{4})\bar{0}44\bar{0}$
$= (0,2,4)(\bar{0},\bar{2},\bar{4})\bar{0}\bar{4}\bar{0}$ □

Since, $N^{(\bar{2}\bar{4}\bar{2})} \geq N^{\bar{2}\bar{4}\bar{2}} =< (1,3,0)(\bar{1},\bar{3},\bar{0}) >$
and $N(\bar{t_2}\bar{t_4}\bar{t_2})^{\{(1,3)(2,0)(\bar{1},\bar{3})(\bar{2},\bar{0})\}} = N\bar{t_0}\bar{t_4}\bar{t_0} \Rightarrow (1,3)(2,0)(\bar{1},\bar{3})(\bar{2},\bar{0}) \in N^{(\bar{2}\bar{4}\bar{2})}$

$\Rightarrow N^{(\bar{2}\bar{4}\bar{2})} =< (1,3,0)(\bar{1},\bar{3},\bar{0}), (1,3)(2,0)(\bar{1},\bar{3})(\bar{2},\bar{0}) >$
$= \{e, (1,3,0)(\bar{1},\bar{3},\bar{0}), (1,0,3)(\bar{1},\bar{0},\bar{3}), (1,2,0)(\bar{1},\bar{2},\bar{0}), (2,3,0)(\bar{2},\bar{3},\bar{0}),$
$(1,3)(2,0)(\bar{1},\bar{3})(\bar{2},\bar{0}), (1,0,2)(\bar{1},\bar{0},\bar{2}), (1,2,3)(\bar{1},\bar{2},\bar{3}), (2,0,3)(\bar{2},\bar{0},\bar{3}),$
$(1,3,2)(\bar{1},\bar{3},\bar{2}), (1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3}), (1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0})\}.$

Therefore, the number of single cosets in the double coset $N\bar{t_2}\bar{t_4}\bar{t_2}N$ are at most

$\frac{|N|}{|N^{(\bar{2}\bar{4}\bar{2})}|} = \frac{5!}{12} = \frac{120}{12} = 10$. The single coset in this double coset has four different names.

Now, take $(N\bar{t_2}\bar{t_4}\bar{t_2})$ conjugates with the elements of $N^{(\bar{2}\bar{4}\bar{2})}$ we obtain the following relation:

$$\bar{2}\bar{4}\bar{2} \sim \bar{0}\bar{4}\bar{0} \sim \bar{3}\bar{4}\bar{3} \sim \bar{1}\bar{4}\bar{1}$$

We can identity the ten single cosets by conjugating $\bar{2}\bar{4}\bar{2} \sim \bar{0}\bar{4}\bar{0} \sim \bar{3}\bar{4}\bar{3} \sim \bar{1}\bar{4}\bar{1}$ with elements in the transversal of $N^{(\bar{2}\bar{4}\bar{2})}$. Hence, the ten distinct single cosets are:

$$\begin{gathered}
\bar{2}\bar{4}\bar{2} \sim \bar{0}\bar{4}\bar{0} \sim \bar{3}\bar{4}\bar{3} \sim \bar{1}\bar{4}\bar{1}\\
\bar{3}\bar{0}\bar{3} \sim \bar{1}\bar{0}\bar{1} \sim \bar{4}\bar{0}\bar{4} \sim \bar{2}\bar{0}\bar{2}\\
232 \sim 030 \sim 434 \sim 131\\
\bar{4}\bar{1}\bar{4} \sim \bar{2}\bar{1}\bar{2} \sim \bar{0}\bar{1}\bar{0} \sim \bar{3}\bar{1}\bar{3}\\
404 \sim 101 \sim 303 \sim 202\\
343 \sim 141 \sim 040 \sim 242\\
\bar{1}\bar{2}\bar{1} \sim \bar{3}\bar{2}\bar{3} \sim \bar{4}\bar{2}\bar{4} \sim \bar{0}\bar{2}\bar{0}\\
010 \sim 212 \sim 414 \sim 313\\
\bar{1}\bar{3}\bar{1} \sim \bar{4}\bar{3}\bar{4} \sim \bar{2}\bar{3}\bar{2} \sim \bar{0}\bar{3}\bar{0}\\
121 \sim 424 \sim 323 \sim 020
\end{gathered}$$

The orbits of $N^{(\bar{2}\bar{4}\bar{2})}$ over $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{4\}$, $\{\bar{4}\}$, $\{1,2,3,0\}$ and $\{\bar{1},\bar{2},\bar{3},\bar{0}\}$. Take representatives 4, $\bar{4}$, 2 and $\bar{2}$ from each orbit, and determine which double coset $N\bar{t_2}\bar{t_4}\bar{t_2}t_4$, $N\bar{t_2}\bar{t_4}\bar{t_2}\bar{t_4}$, $N\bar{t_2}\bar{t_4}\bar{t_2}t_2$ and $N\bar{t_2}\bar{t_4}\bar{t_2}\bar{t_2}$ belong. However,

$N\bar{t_2}\bar{t_4}\bar{t_2}t_4 \in [\bar{2}424]$ **(By Lemma-VIII)**

$N\bar{t_2}\bar{t_4}\bar{t_2}\bar{t_4} \in N\bar{t_2}\bar{t_4}\bar{t_2}\bar{t_4}N$ (New double coset denote by $[\bar{2}\bar{4}\bar{2}\bar{4}]$

$N\bar{t_2}\bar{t_4}\bar{t_2}t_2 = N\bar{t_2}\bar{t_4} \in [\bar{2}\bar{4}]$

$N\bar{t_2}\bar{t_4}\bar{t_2}\bar{t_2} = \bar{t_2}\bar{t_4}t_2 \in [\bar{2}\bar{4}2]$

Therefore, four symmetric generators go back to $[\bar{2}\bar{4}]$; four go to $[\bar{2}\bar{4}2]$; one goes to $[\bar{2}424]$;

one extends to new double coset $[\bar{2}\bar{4}\bar{2}\bar{4}]$

Consider the next double coset $[\bar{2}\bar{4}\bar{3}]$.

Claim: $\bar{2}\bar{4}\bar{3} \sim \bar{2}\bar{4}\bar{1} \sim 423$

Proof. By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$ conjugating with the elements of N, we obtain the following relations:

1. $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}} = (\bar{2}3)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}}$

$\Rightarrow (2,4,3)(\bar{2},\bar{4},\bar{3})2\bar{3} = 2\bar{4}.$

$\Rightarrow \bar{2}\bar{4}\bar{3} = 2\underline{2\bar{4}}\bar{3} = 2(2,4,3)(\bar{2},\bar{4},\bar{3})2\bar{3}\bar{3} = (2,4,3)(\bar{2},\bar{4},\bar{3})423.$

2. $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{2},4,\bar{3})(2,\bar{4},3,\bar{1})(0,\bar{0})\}} = (\bar{2}3)^{\{(1,\bar{2},4,\bar{3})(2,\bar{4},3,\bar{1})(0,\bar{0})\}}$

$\Rightarrow (4,3,1)(\bar{4},\bar{3},\bar{1})4\bar{1} = 4\bar{3}$

$\Rightarrow \bar{2}\bar{4}\bar{3} = \bar{2}4\underline{4\bar{3}} = \bar{2}4(4,3,1)(\bar{4},\bar{3},\bar{1})4\bar{1} = \bar{2}34\bar{1} = (2,3,4)(\bar{2}\bar{3}\bar{4})\bar{2}44\bar{1}$

$= (2,3,4)(\bar{2}\bar{3}\bar{4})\bar{2}4\bar{1}.$ □

Since, $N^{(\bar{2}\bar{4}\bar{3})} \geq N^{\bar{2}\bar{4}\bar{3}} =< e >$,and

$N(\bar{t_2}\bar{t_4}\bar{t_3})^{\{(1,\bar{1})(2,\bar{4})(3,\bar{3})(4,\bar{2})(0,\bar{0})\}} = Nt_4t_2t_3 \Rightarrow (1,\bar{1})(2,\bar{4})(3,\bar{3})(4,\bar{2})(0,\bar{0}) \in N^{(\bar{2}\bar{4}\bar{3})}$

$N(\bar{t_2}\bar{t_4}\bar{t_3})^{\{(1,0,3)(\bar{1},\bar{0},\bar{3})\}} = \bar{t_2}\bar{t_4}\bar{t_1} \Rightarrow (1,0,3)(\bar{1},\bar{0},\bar{3}) \in N^{(\bar{2}\bar{4}\bar{3})}$

$\Rightarrow N^{(\bar{2}\bar{4}\bar{3})} =< (1,\bar{1})(2,\bar{4})(3,\bar{3})(4,\bar{2})(0,\bar{0}), (1,0,3)(\bar{1},\bar{0},\bar{3}) >$

$= \{e, (1,\bar{1})(2,\bar{4})(3,\bar{3})(4,\bar{2})(0,\bar{0}), (1,\bar{0},3,\bar{1},0,\bar{3})(2,\bar{4})(4,\bar{2}), (1,\bar{3},0,\bar{1},3,\bar{0})(2,\bar{4})(4,\bar{2}),$
$(1,0,3)(\bar{1},\bar{0},\bar{3}), (1,3,0)(\bar{1},\bar{3},\bar{0})\}$

Now, again, take $(N\bar{t_2}\bar{t_4}\bar{t_3})$ conjugates with the elements of $N^{(\bar{2}\bar{4}\bar{3})}$ we obtain the following relation:

$$\bar{2}\bar{4}\bar{3} \sim \bar{2}\bar{4}\bar{1} \sim \bar{2}\bar{4}\bar{0} \sim 420 \sim 421 \sim 423$$

Therefore, the number of single cosets in the double coset $N\bar{t_2}\bar{t_4}\bar{t_3}N$ are at most

$\frac{|N|}{|N^{(\bar{2}\bar{4}\bar{3})}|} = \frac{5!}{6} = \frac{120}{6} = 20$. Each of single coset in this double coset has six different names.

Again, we can identity all the twenty single cosets in $[\bar{2}\bar{4}\bar{3}]$ by conjugating ($\bar{2}\bar{4}\bar{3} \sim \bar{2}\bar{4}\bar{1} \sim \bar{2}\bar{4}\bar{0} \sim 420 \sim 421 \sim 423$) with elements in the transversal of $N^{(\bar{2}\bar{4}\bar{3})}$. Hence, the twenty distinct single cosets are:

1. $\bar{2}\bar{4}\bar{3} \sim \bar{2}\bar{4}\bar{1} \sim \bar{2}\bar{4}\bar{0} \sim 420 \sim 421 \sim 423$

2. $\bar{3}\bar{0}\bar{1} \sim \bar{3}\bar{0}\bar{4} \sim \bar{3}\bar{0}\bar{2} \sim 032 \sim 034 \sim 031$

3. $234 \sim 231 \sim 230 \sim \bar{3}\bar{2}\bar{0} \sim \bar{3}\bar{2}\bar{1} \sim \bar{3}\bar{2}\bar{4}$

4. $\bar{4}\bar{1}\bar{2} \sim \bar{4}\bar{1}\bar{0} \sim \bar{4}\bar{1}\bar{3} \sim 143 \sim 140 \sim 142$

5. $401 \sim 403 \sim 402 \sim \bar{0}\bar{4}\bar{2} \sim \bar{0}\bar{4}\bar{3} \sim \bar{0}\bar{4}\bar{1}$

6. $340 \sim 342 \sim 341 \sim \bar{4}\bar{3}\bar{1} \sim \bar{4}\bar{3}\bar{2} \sim \bar{4}\bar{3}\bar{0}$

7. $\bar{0}\bar{2}\bar{3} \sim \bar{0}\bar{2}\bar{1} \sim \bar{0}\bar{2}\bar{4} \sim 204 \sim 201 \sim 203$

8. $\bar{1}\bar{3}\bar{2} \sim \bar{1}\bar{3}\bar{0} \sim \bar{1}\bar{3}\bar{4} \sim 314 \sim 310 \sim 312$

9. $012 \sim 014 \sim 013 \sim \bar{1}\bar{0}\bar{3} \sim \bar{1}\bar{0}\bar{4} \sim \bar{1}\bar{0}\bar{2}$

10. $024 \sim 021 \sim 023 \sim \bar{2}\bar{0}\bar{3} \sim \bar{2}\bar{0}\bar{1} \sim \bar{2}\bar{0}\bar{4}$

11. $123 \sim 120 \sim 124 \sim \bar{2}\bar{1}\bar{4} \sim \bar{2}\bar{1}\bar{0} \sim \bar{2}\bar{1}\bar{3}$

12. $\bar{0}\bar{1}\bar{2} \sim \bar{0}\bar{1}\bar{3} \sim \bar{0}\bar{1}\bar{4} \sim 104 \sim 103 \sim 102$

13. $134 \sim 130 \sim 132 \sim \bar{3}\bar{1}\bar{2} \sim \bar{3}\bar{1}\bar{0} \sim \bar{3}\bar{1}\bar{4}$

14. $\bar{1}\bar{2}\bar{4} \sim \bar{1}\bar{2}\bar{0} \sim \bar{1}\bar{2}\bar{3} \sim 213 \sim 210 \sim 214$

15. $240 \sim 241 \sim 243 \sim \bar{4}\bar{2}\bar{3} \sim \bar{4}\bar{2}\bar{1} \sim \bar{4}\bar{2}\bar{0}$

16. $\bar{1}\bar{4}\bar{3} \sim \bar{1}\bar{4}\bar{0} \sim \bar{1}\bar{4}\bar{2} \sim 412 \sim 410 \sim 413$

17. $\bar{2}\bar{3}\bar{0} \sim \bar{2}\bar{3}\bar{1} \sim \bar{2}\bar{3}\bar{4} \sim 324 \sim 321 \sim 320$

18. $301 \sim 302 \sim 304 \sim \bar{0}\bar{3}\bar{4} \sim \bar{0}\bar{3}\bar{2} \sim \bar{0}\bar{3}\bar{1}$

19. $\bar{3}\bar{4}\bar{0} \sim \bar{3}\bar{4}\bar{1} \sim \bar{3}\bar{4}\bar{2} \sim 432 \sim 431 \sim 430$

20. $\bar{4}\bar{0}\bar{1} \sim \bar{4}\bar{0}\bar{2} \sim \bar{4}\bar{0}\bar{3} \sim 043 \sim 042 \sim 041$

The orbits of $N^{(\bar{2}\bar{4}\bar{3})}$ over $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{2,\bar{4}\}$, $\{4,\bar{2}\}$, and $\{(1,3,0,\bar{1},\bar{3},\bar{0}\}$. We now take representatives $\bar{4}$, 4 and 3 from each orbit, then determine which double coset $N\bar{t}_2\bar{t}_4\bar{t}_3t_4$, $N\bar{t}_2\bar{t}_4\bar{t}_3\bar{t}_4$ and $N\bar{t}_2\bar{t}_4\bar{t}_3t_3$ belong. However,
$N\bar{t}_2\bar{t}_4\bar{t}_3\bar{t}_4 \in [\bar{2}42]$ **(By Lemma-VI)**

$N\bar{t}_2\bar{t}_4\bar{t}_3t_4 \in [\bar{2}424]$ **(By Lemma-VII)**
$N\bar{t}_2\bar{t}_4\bar{t}_3t_3 = N\bar{t}_2\bar{t}_4 \in [\bar{2}\bar{4}]$
Therefore, two symmetric generators go to $[\bar{2}42]$; two go to $[\bar{2}424]$; the other six go back to $[\bar{2}\bar{4}]$ and none of them advance.

Next, consider the double coset $[\bar{2}\bar{4}\bar{2}\bar{4}]$.
Claim: $\bar{2}\bar{4}\bar{2}\bar{4} \sim \bar{4}\bar{1}\bar{4}\bar{1}$

Proof. By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$ conjugating with the elements of N, we obtain the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{0},3,\bar{4})(2,\bar{2})(4,\bar{1},0,\bar{3})\}} = (\bar{2}3)^{\{(1,\bar{0},3,\bar{4})(2,\bar{2})(4,\bar{1},0,\bar{3})\}}$
  $\Rightarrow (\bar{2},\bar{4},\bar{1})(2,4,1)2\bar{1} = 2\bar{4}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,0,3,4,2)(\bar{1}\bar{0}\bar{3}\bar{4}\bar{2})\}} = (\bar{2}3)^{\{(1,0,3,4,2)(\bar{1}\bar{0}\bar{3}\bar{4}\bar{2})\}}$
  $\Rightarrow (1,4,2)(\bar{1},\bar{1},\bar{1})\bar{1}2 = \bar{1}4.$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{0},3,\bar{2})(2,\bar{1},0,\bar{3})(4,\bar{4})\}} = (\bar{2}3)^{\{(1,\bar{0},3,\bar{2})(2,\bar{1},0,\bar{3})(4,\bar{4})\}}$
  $\Rightarrow (\bar{1},\bar{2},\bar{4})(1,2,4)1\bar{4} = 1\bar{2}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,0,3,2,4)(\bar{1}\bar{0}\bar{3}\bar{2}\bar{4})\}} = (\bar{2}3)^{\{(1,0,3,2,4)(\bar{1}\bar{0}\bar{3}\bar{2}\bar{4})\}}$
  $\Rightarrow (4,2,1)(\bar{4},\bar{2},\bar{1})\bar{4}1 = \bar{4}2$

From the relations above,
$\Rightarrow \bar{2}\bar{4}\underline{\bar{2}}\bar{4} = \bar{2}\bar{4}2\underline{\bar{2}\bar{4}} = \bar{2}\bar{4}2(\bar{2},\bar{4},\bar{1})(2,4,1)2\bar{1} = \bar{4}\underline{\bar{1}4}2\bar{1} = \bar{4}(1,4,2)(\bar{1},\bar{1},\bar{1})\bar{1}22\bar{1}$
$= \bar{2}\bar{1}2\bar{1} = \bar{2}1\underline{1\bar{2}}\bar{1} = \bar{2}1(\bar{1},\bar{2},\bar{4})(1,2,4)1\bar{4}\bar{1} = \underline{\bar{4}2}1\bar{4}\bar{1} = (4,2,1)(\bar{4},\bar{2},\bar{1})\bar{4}11\bar{4}\bar{1}$
$= (4,2,1)(\bar{4},\bar{2},\bar{1})\bar{4}\bar{1}\bar{4}\bar{1}.$ ☐

Since, $N^{(\bar{2}\bar{4}\bar{2}\bar{4})} \geq N^{\bar{2}\bar{4}\bar{2}\bar{4}} =< (1,3,0)(\bar{1}\bar{3}\bar{0}) >$
and $N(\bar{t}_2\bar{t}_4\bar{t}_2\bar{t}_4)^{\{(1,3,0,2,4)(\bar{1},\bar{3},\bar{0},\bar{2},\bar{4})\}} = N\bar{t}_2\bar{t}_4\bar{t}_2\bar{t}_4 \Rightarrow (1,3,0,2,4)(\bar{1},\bar{3},\bar{0},\bar{2},\bar{4}) \in N^{(\bar{2}\bar{4}\bar{2}\bar{4})}$
$\Rightarrow N^{(\bar{2}\bar{4}\bar{2}\bar{4})} =< (1,3,0)(\bar{1},\bar{3},\bar{0}), (1,3,0,2,4)(\bar{1},\bar{3},\bar{0},\bar{2},\bar{4}) >$
$= \{e, (1,3,0)(\bar{1},\bar{3},\bar{0}), (1,0,3)(\bar{1},\bar{0},\bar{3}), (1,4,2,0,3)(\bar{1},\bar{4},\bar{2},\bar{0},\bar{3}), (1,2,4)(\bar{1},\bar{2},\bar{4}),$
$(2,4,0)(\bar{2},\bar{4},\bar{0}), (1,3,0,2,4)(\bar{1},\bar{3},\bar{0},\bar{2},\bar{4}), (1,0,2,4,3)(\bar{1},\bar{0},\bar{2},\bar{4},\bar{3}), (1,4,3)(\bar{1},\bar{4},\bar{3}),$
$(1,2)(4,0)(\bar{1},\bar{2})(\bar{4},\bar{0}), (2,0,4)(\bar{2},\bar{0},\bar{4}), (1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2}), (1,0,4,2,3)(\bar{1},\bar{0},\bar{4},\bar{2},\bar{3}),$
$(1,4,0,2,3)(\bar{1},\bar{4},\bar{0},\bar{2},\bar{3}), (1,2,0)(\bar{1},\bar{2},\bar{0}), (2,3,0)(\bar{2},\bar{3},\bar{0}), (1,3)(2,0)(\bar{1},\bar{3})(\bar{2},\bar{0}),$
$(1,0,2)(\bar{1},\bar{0},\bar{2}), (1,4,2)(\bar{1},\bar{4},\bar{2}), (1,2,3,0,4)(\bar{1},\bar{2},\bar{3},\bar{0},\bar{4}), (3,0,4)(\bar{3},\bar{0},\bar{4}),$

$(1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0}),(1,0,4)(\bar{1},\bar{0},\bar{4}),(1,4)(2,0)(\bar{1},\bar{4})(\bar{2},\bar{0}),(1,2,4,3,0)(\bar{1},\bar{2},\bar{4},\bar{3},\bar{0}),$
$(2,4,3)(\bar{2},\bar{4},\bar{3}),(1,3,2,4,0)(\bar{1},\bar{3},\bar{2},\bar{4},\bar{0}),(1,0,3,2,4)(\bar{1},\bar{0},\bar{3},\bar{2},\bar{4}),(1,4)(3,0)(\bar{1},\bar{4})(\bar{3},\bar{0}),$
$(1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4}),(2,0)(3,4)(\bar{2},\bar{0})(\bar{3},\bar{4}),(1,3,4,0,2)(\bar{1},\bar{3},\bar{4},\bar{0},\bar{2}),$
$(1,0,2,3,4)(\bar{1},\bar{0},\bar{2},\bar{3},\bar{4}),(1,4)(2,3)(\bar{1},\bar{4})(\bar{2},\bar{3}),(1,2,0,4,3)(\bar{1},\bar{2},\bar{0},\bar{4},\bar{3}),$
$(2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0}),(1,3,2,0,4)(\bar{1},\bar{3},\bar{2},\bar{0},\bar{4}),(1,0,4,3,2)(\bar{1},\bar{0},\bar{4},\bar{3},\bar{2}),$
$(1,4,3,0,2)(\bar{1},\bar{4},\bar{3},\bar{0},\bar{2}),(1,2,3,4,0)(\bar{1},\bar{2},\bar{3},\bar{4},\bar{0}),(3,4,0)(\bar{3},\bar{4},\bar{0}),$
$(1,3,4)(\bar{1},\bar{3},\bar{4}),(1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4}),(1,4,3,2,0)(\bar{1},\bar{4},\bar{3},\bar{2},\bar{0}),(1,2,4,0,3)(\bar{1},\bar{2},\bar{4},\bar{0},\bar{3}),$
$(2,4)(3,0)(\bar{2},\bar{4})(\bar{3},\bar{0}),(1,3)(2,4)(\bar{1},\bar{3})(\bar{2},\bar{4}),(1,0)(2,4)(\bar{1},\bar{0})(\bar{2},\bar{4}),(1,4,0)(\bar{1},\bar{4},\bar{0}),$
$(1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0}),(2,0,3)(\bar{2},\bar{0},\bar{3}),(1,3,2)(\bar{1},\bar{3},\bar{2}),(1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3}),$
$(1,4,2,3,0)(\bar{1},\bar{4},\bar{2},\bar{3},\bar{0}),(1,2,0,3,4)(\bar{1},\bar{2},\bar{0},\bar{3},\bar{4}),(2,3,4)(\bar{2},\bar{3},\bar{4}),$
$(1,3,4,2,0)(\bar{1},\bar{3},\bar{4},\bar{2},\bar{0}),(1,0,3,4,2)(\bar{1},\bar{0},\bar{3},\bar{4},\bar{2}),(1,4,0,3,2)(\bar{1},\bar{4},\bar{0},\bar{3},\bar{2}),$
$(1,2,3)(\bar{1},\bar{2},\bar{3})\}.$

We now take $(N\bar{t_2}\bar{t_4}\bar{t_2}\bar{t_4})$ conjugates with elements of $N^{(\bar{2}\bar{4}\bar{2}\bar{4})}$, we obtain the following relations:

$\bar{4}\bar{1}\bar{4}\bar{1} \sim \bar{0}\bar{3}\bar{0}\bar{3} \sim \bar{2}\bar{3}\bar{2}\bar{3} \sim \bar{4}\bar{3}\bar{4}\bar{3} \sim \bar{2}\bar{0}\bar{2}\bar{0} \sim \bar{0}\bar{2}\bar{0}\bar{2} \sim \bar{3}\bar{0}\bar{3}\bar{0} \sim \bar{3}\bar{1}\bar{3}\bar{1} \sim \bar{1}\bar{0}\bar{1}\bar{0} \sim \bar{1}\bar{3}\bar{1}\bar{3} \sim \bar{0}\bar{1}\bar{0}\bar{1} \sim \bar{4}\bar{0}\bar{4}\bar{0} \sim \bar{1}\bar{2}\bar{1}\bar{2} \sim \bar{4}\bar{2}\bar{4}\bar{2} \sim \bar{0}\bar{4}\bar{0}\bar{4} \sim \bar{1}\bar{4}\bar{1}\bar{4} \sim \bar{3}\bar{2}\bar{3}\bar{2} \sim \bar{2}\bar{4}\bar{2}\bar{4} \sim \bar{3}\bar{4}\bar{3}\bar{4} \sim \bar{2}\bar{1}\bar{2}\bar{1}.$

Therefore, the number of single cosets in the double coset $N\bar{t_2}\bar{t_4}\bar{t_3}N$ are at most

$\frac{|N|}{|N^{(2424)}|} = \frac{5!}{60} = \frac{120}{60} = 2$. Each single coset has twenty different names.

From the relations, we can identity all the single cosets in the double coset $[\bar{2}\bar{4}\bar{2}\bar{4}]$ by conjugating

$(\bar{4}\bar{1}\bar{4}\bar{1} \sim \bar{0}\bar{3}\bar{0}\bar{3} \sim \bar{2}\bar{3}\bar{2}\bar{3} \sim \bar{4}\bar{3}\bar{4}\bar{3} \sim \bar{2}\bar{0}\bar{2}\bar{0} \sim \bar{0}\bar{2}\bar{0}\bar{2} \sim \bar{3}\bar{0}\bar{3}\bar{0} \sim \bar{3}\bar{1}\bar{3}\bar{1} \sim \bar{1}\bar{0}\bar{1}\bar{0} \sim \bar{1}\bar{3}\bar{1}\bar{3} \sim \bar{0}\bar{1}\bar{0}\bar{1} \sim \bar{4}\bar{0}\bar{4}\bar{0} \sim \bar{1}\bar{2}\bar{1}\bar{2} \sim \bar{4}\bar{2}\bar{4}\bar{2} \sim \bar{0}\bar{4}\bar{0}\bar{4} \sim \bar{1}\bar{4}\bar{1}\bar{4} \sim \bar{3}\bar{2}\bar{3}\bar{2} \sim \bar{2}\bar{4}\bar{2}\bar{4} \sim \bar{3}\bar{4}\bar{3}\bar{4} \sim \bar{2}\bar{1}\bar{2}\bar{1})$

with elements in the transversal of $N^{(\bar{2}\bar{4}\bar{2}\bar{4})}$. Hence, the two distinct single cosets are:

1. $\bar{4}\bar{1}\bar{4}\bar{1} \sim \bar{0}\bar{3}\bar{0}\bar{3} \sim \bar{2}\bar{3}\bar{2}\bar{3} \sim \bar{4}\bar{3}\bar{4}\bar{3} \sim \bar{2}\bar{0}\bar{2}\bar{0} \sim \bar{0}\bar{2}\bar{0}\bar{2} \sim \bar{3}\bar{0}\bar{3}\bar{0} \sim \bar{3}\bar{1}\bar{3}\bar{1} \sim \bar{1}\bar{0}\bar{1}\bar{0} \sim \bar{1}\bar{3}\bar{1}\bar{3} \sim \bar{0}\bar{1}\bar{0}\bar{1} \sim \bar{4}\bar{0}\bar{4}\bar{0} \sim \bar{1}\bar{2}\bar{1}\bar{2} \sim \bar{4}\bar{2}\bar{4}\bar{2} \sim \bar{0}\bar{4}\bar{0}\bar{4} \sim \bar{1}\bar{4}\bar{1}\bar{4} \sim \bar{3}\bar{2}\bar{3}\bar{2} \sim \bar{2}\bar{4}\bar{2}\bar{4} \sim \bar{3}\bar{4}\bar{3}\bar{4} \sim \bar{2}\bar{1}\bar{2}\bar{1}.$

2. $3131 \sim 0404 \sim 2424 \sim 3434 \sim 2020 \sim 0202 \sim 4040 \sim 4141 \sim 1010 \sim 1414 \sim 0101 \sim 3030 \sim 1212 \sim 3232 \sim 0303 \sim 1313 \sim 4242 \sim 2323 \sim 4343 \sim 2121.$

The Orbits of $N^{(\bar{2}\bar{4}\bar{2}\bar{4})}$ over $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{1,2,3,4,0\}$, and $\{(\bar{1},\bar{2},\bar{3},\bar{4},\bar{0}\}$. Take a representative from each orbit namely $\{4\}$ and $\{\bar{4}\}$, then determine which double

coset $N\bar{t_2}\bar{t_4}\bar{t_2}\bar{t_4}4$ and $N\bar{t_2}\bar{t_4}\bar{t_2}\bar{t_4}\bar{t_4}$ belong. However,
$N\bar{t_2}\bar{t_4}\bar{t_2}\bar{t_4}4 = N\bar{t_2}\bar{t_4}\bar{t_2} \in [\bar{2}\bar{4}\bar{2}]$
$N\bar{t_2}\bar{t_4}\bar{t_2}\bar{t_4}\bar{t_4} = \bar{t_2}\bar{t_4}\bar{t_2}t_4 \in [\bar{2}424]$ **(By Lemma-VIII)**

Since five symmetric generators go back to $[\bar{2}\bar{4}\bar{2}]$, and the other five go to $[\bar{2}424]$. Therefore, no symmetric generator advances.

Next, consider the double coset $[\bar{2}4\bar{2}]$.
Claim: $\bar{2}4\bar{2} \sim \bar{0}2\bar{0}$

Proof. By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$ conjugating with an elements of N, we obtain the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{1})(2,\bar{4},0,\bar{3})(3,\bar{2},4,\bar{0})\}} = (\bar{2}3)^{\{1,\bar{1})(2,\bar{4},0,\bar{3})(3,\bar{2},4\bar{0})\}}$
  $\Rightarrow (\bar{4},\bar{2},\bar{0})(4,2,0)4\bar{0} = 4\bar{2}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,3,4,2,0)(\bar{1},\bar{3},\bar{4},\bar{2},\bar{0})\}} = (\bar{2}3)^{\{(1,3,4,2,0)(\bar{1},\bar{3},\bar{4},\bar{2},\bar{0})\}}$
  $\Rightarrow (0,4,2)(\bar{0},\bar{4},\bar{2})\bar{0}2 = \bar{0}4$

From the relations above, we have
$\bar{2}\underline{4\bar{2}} = \bar{2}(\bar{4},\bar{2},\bar{0})(4,2,0)4\bar{0} = (\bar{4},\bar{2},\bar{0})(4,2,0)\bar{0}4\bar{0}$
$= (\bar{4},\bar{2},\bar{0})(4,2,0)(0,4,2)(\bar{0},\bar{4},\bar{2})\bar{0}2\bar{0} = (\bar{4},\bar{0},\bar{2})(4,0,2)\bar{0}2\bar{0}.$ □

Since, $N^{(\bar{2}4\bar{2})} \geq N^{\bar{2}4\bar{2}} =< (1,3,0)(\bar{1},\bar{3},\bar{0}) >$
and $N(\bar{t_2}t_4\bar{t_2})^{\{(1,4,2,0,3)(\bar{1},\bar{4},\bar{2},\bar{0},\bar{3})\}} = N\bar{t_0}t_2\bar{t_0} \Rightarrow (1,4,2,0,3)(\bar{1},\bar{4},\bar{2},\bar{0},\bar{3}) \in N^{(\bar{2}4\bar{2})}$
$\Rightarrow N^{(\bar{2}4\bar{2})} =< (1,3,0)(\bar{1},\bar{3},\bar{0}),(1,4,2,0,3)(\bar{1},\bar{4},\bar{2},\bar{0},\bar{3}) >$
$= \{e, (1,3,0)(\bar{1},\bar{3},\bar{0}), (1,4,2,0,3)(\bar{1},\bar{4},\bar{2},\bar{0},\bar{3}), (1,0,3)(\bar{1},\bar{0},\bar{3}), (1,2,3,4,0)(\bar{1},\bar{2},\bar{3},\bar{4},\bar{0}),$
$(2,0,4)(\bar{2},\bar{0},\bar{4}), (1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2}), (1,4,0,2,3)(\bar{1},\bar{4},\bar{0},\bar{2},\bar{3}),$
$(1,0,4,2,3)(\bar{1},\bar{0},\bar{4},\bar{2},\bar{3}), (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4}), (2,4,0)(\bar{2},\bar{4},\bar{0}), (1,3,0,2,4)(\bar{1},\bar{3},\bar{0},\bar{2},\bar{4}),$
$(1,4,3)(\bar{1},\bar{4},\bar{3}), (1,0,2,4,3)(\bar{1},\bar{0},\bar{2},\bar{4},\bar{3}), (1,2,0,3,4)(\bar{1},\bar{2},\bar{0},\bar{3},\bar{4}), (2,3,0)(\bar{2},\bar{3},\bar{0}),$
$(1,3)(2,0)(\bar{1},\bar{3})(\bar{2},\bar{0}), (1,4,2)(\bar{1},\bar{4},\bar{2}), (1,0,2)(\bar{1},\bar{0},\bar{2}), (1,2,4,0,3)(\bar{1},\bar{2},\bar{4},\bar{0},\bar{3}),$
$(3,0,4)(\bar{3},\bar{0},\bar{4}), (1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0}), (1,4)(2,0)(\bar{1},\bar{4})(\bar{2},\bar{0}), (1,0,4)(\bar{1},\bar{0},\bar{4}),$
$(1,2,3)(\bar{1},\bar{2},\bar{3}), (2,0)(3,4)(\bar{2},\bar{0})(\bar{3},\bar{4}), (1,3,4,0,2)(\bar{1},\bar{3},\bar{4},\bar{0},\bar{2}), (1,4)(2,3)(\bar{1},\bar{4})(\bar{2},\bar{3}),$
$(1,0,2,3,4)(\bar{1},\bar{0},\bar{2},\bar{3},\bar{4}), (1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0}), (2,4,3)(\bar{2},\bar{4},\bar{3}), (1,3,2,4,0)(\bar{1},\bar{3},\bar{2},\bar{4},\bar{0}),$
$(1,4)(3,0)(\bar{1},\bar{4})(\bar{3},\bar{0}), (1,0,3,2,4)(\bar{1},\bar{0},\bar{3},\bar{2},\bar{4}), (1,2,0)(\bar{1},\bar{2},\bar{0}), (2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0}),$

$(1,3,2,0,4)(\bar{1},\bar{3},\bar{2},\bar{0},\bar{4}),(1,4,3,0,2)(\bar{1},\bar{4},\bar{3},\bar{0},\bar{2}),(1,0,4,3,2)(\bar{1},\bar{0},\bar{4},\bar{3},\bar{2}),$
$(1,2,4)(\bar{1},\bar{2},\bar{4}),(3,4,0)(\bar{3},\bar{4},\bar{0}),(1,3,4)(\bar{1},\bar{3},\bar{4}),(1,4,3,2,0)(\bar{1},\bar{4},\bar{3},\bar{2},\bar{0}),$
$(1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4}),(1,2,3,0,4)(\bar{1},\bar{2},\bar{3},\bar{0},\bar{4}),(2,0,3)(\bar{2},\bar{0},\bar{3}),(1,3,2)(\bar{1},\bar{3},\bar{2}),$
$(1,4,2,3,0)(\bar{1},\bar{4},\bar{2},\bar{3},\bar{0}),(1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3}),(1,2)(4,0)(\bar{1},\bar{2})(\bar{4},\bar{0}),(2,4)(3,0)(\bar{2},\bar{4})(\bar{3},\bar{0}),$
$(1,3)(2,4)(\bar{1},\bar{3})(\bar{2},\bar{4}),(1,4,0)(\bar{1},\bar{4},\bar{0}),(1,0)(2,4)(\bar{1},\bar{0})(\bar{2},\bar{4}),(1,2,0,4,3)(\bar{1},\bar{2},\bar{0},\bar{4},\bar{3}),$
$(2,3,4)(\bar{2},\bar{3},\bar{4}),(1,3,4,2,0)(\bar{1},\bar{3},\bar{4},\bar{2},\bar{0}),(1,4,0,3,2)(\bar{1},\bar{4},\bar{0},\bar{3},\bar{2}),$
$(1,0,3,4,2)(\bar{1},\bar{0},\bar{3},\bar{4},\bar{2}),(1,2,4,3,0)(\bar{1},\bar{2},\bar{4},\bar{3},\bar{0})\}.$

Take $(N\bar{t_2}t_4\bar{t_2})$ conjugates with elements of $N^{(\bar{2}4\bar{2})}$, we obtain the following relations:
$\bar{0}4\bar{0} \sim \bar{1}2\bar{1} \sim \bar{2}1\bar{2} \sim \bar{4}2\bar{4} \sim \bar{2}4\bar{2} \sim \bar{2}0\bar{2} \sim \bar{1}3\bar{1} \sim \bar{1}0\bar{1} \sim \bar{3}2\bar{3} \sim \bar{0}1\bar{0} \sim \bar{0}2\bar{0} \sim \bar{4}1\bar{4} \sim \bar{0}3\bar{0} \sim \bar{3}1\bar{3} \sim \bar{2}3\bar{2} \sim \bar{3}4\bar{3} \sim \bar{3}0\bar{3} \sim \bar{1}4\bar{1} \sim \bar{4}3\bar{4} \sim \bar{4}0\bar{4}.$

Therefore, the number of single cosets in the double coset $N\bar{t_2}t_4\bar{t_2}N$ are at most

$\frac{|N|}{|N^{(\bar{2}4\bar{2})}|} = \frac{5!}{60} = \frac{120}{60} = 2$. Each single coset has twenty different names.

From the relations above, we can identify all the single cosets in double coset $[\bar{2}4\bar{2}]$by cojugating
$\bar{0}4\bar{0} \sim \bar{1}2\bar{1} \sim \bar{2}1\bar{2} \sim \bar{4}2\bar{4} \sim \bar{2}4\bar{2} \sim \bar{2}0\bar{2} \sim \bar{1}3\bar{1} \sim \bar{1}0\bar{1} \sim \bar{3}2\bar{3} \sim \bar{0}1\bar{0} \sim \bar{0}2\bar{0} \sim \bar{4}1\bar{4} \sim \bar{0}3\bar{0} \sim \bar{3}1\bar{3} \sim \bar{2}3\bar{2} \sim \bar{3}4\bar{3} \sim \bar{3}0\bar{3} \sim \bar{1}4\bar{1} \sim \bar{4}3\bar{4} \sim \bar{4}0\bar{4}$
with elements of the transversal of $N^{(\bar{2}4\bar{2})}$. Hence, the two single cosets of double coset $[\bar{2}4\bar{2}]$ are:

1. $\bar{0}4\bar{0} \sim \bar{1}2\bar{1} \sim \bar{2}1\bar{2} \sim \bar{4}2\bar{4} \sim \bar{2}4\bar{2} \sim \bar{2}0\bar{2} \sim \bar{1}3\bar{1} \sim \bar{1}0\bar{1} \sim \bar{3}2\bar{3} \sim \bar{0}1\bar{0} \sim \bar{0}2\bar{0} \sim \bar{4}1\bar{4} \sim \bar{0}3\bar{0} \sim \bar{3}1\bar{3} \sim \bar{2}3\bar{2} \sim \bar{3}4\bar{3} \sim \bar{3}0\bar{3} \sim \bar{1}4\bar{1} \sim \bar{4}3\bar{4} \sim \bar{4}0\bar{4}.$

2. $0\bar{3}0 \sim 1\bar{2}1 \sim 2\bar{1}2 \sim 3\bar{2}3 \sim 2\bar{3}2 \sim 2\bar{0}2 \sim 1\bar{4}1 \sim 1\bar{0}1 \sim 4\bar{2}4 \sim 0\bar{1}0 \sim 0\bar{2}0 \sim 3\bar{1}3 \sim 0\bar{4}0 \sim 4\bar{1}4 \sim 2\bar{4}2 \sim 4\bar{3}4 \sim 4\bar{0}4 \sim 1\bar{3}1 \sim 3\bar{4}3 \sim 3\bar{0}3.$

The orbits of $N^{(\bar{2}4\bar{2})}$ over $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{1,2,3,4,0\}$, and $\{(\bar{1},\bar{2},\bar{3},\bar{4},\bar{0}\}$. We now take the representatives 2 and $\bar{2}$ from each orbit, and determine which double coset $N\bar{t_2}t_4\bar{t_2}t_2$ and $N\bar{t_2}t_4\bar{t_2}\bar{t_2}$ belong. However,
$N\bar{t_2}t_4\bar{t_2}t_2 = N\bar{t_2}t_4 \in [\bar{2}4]$
$N\bar{t_2}t_4\bar{t_2}\bar{t_2} = N\bar{t_2}t_4t_2 \in [\bar{2}42]$
Therefore, five symmetric generators go back to $[\bar{2}4]$; the other five go to $[\bar{2}42]$; and no

symmetric generator advances.

Again, we now consider the double coset $[\bar{2}42]$.

Claim: $\bar{2}42 \sim \bar{2}32$

Proof. By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3 \Rightarrow (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}42 = \bar{2}32$. $\square$

Since, $N^{(\bar{2}42)} \geq N^{bar242} =< (1,3,0)(\bar{2},\bar{2},\bar{2}) >$
and $(N\bar{t_2}t_4t_2)^{\{(1,4,3)(\bar{1},\bar{4},\bar{3})\}} = N\bar{t_2}t_3t_2 \Rightarrow (1,4,3)(\bar{1},\bar{4},\bar{3}) \in N^{(\bar{2}42)}$.
$\Rightarrow N^{(\bar{2}42)} =< (1,3,0)(\bar{1},\bar{3},\bar{0}),(1,4,3)(\bar{1},\bar{4},\bar{3}) >$
$= \{e,(1,3,0)(\bar{1},\bar{3},\bar{0}),(1,4,3)(\bar{1},\bar{4},\bar{3}),(1,0,3)(\bar{1},\bar{0},\bar{3}),(3,0,4)(\bar{3},\bar{0},\bar{4}),(1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0}),$
$(1,4)(3,0)(\bar{1},\bar{4})(\bar{3},\bar{0}),(1,0,4)(\bar{1},\bar{0},\bar{4}),(3,4,0)(\bar{3},\bar{4},\bar{0}),(1,3,4)(\bar{1},\bar{3},\bar{4}),(1,4,0)(\bar{1},\bar{4},\bar{0}),$
$(1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4})\}$.
Now, take $(N\bar{t_2}t_4t_2)$ conjugates with elements of $N^{(\bar{2}42)}$, we obtain the following relations:

$$\bar{2}42 \sim \bar{2}32 \sim \bar{2}12 \sim \bar{2}02$$

Therefore, the number of single cosets in the double coset $[\bar{2}42]$ are at most,

$\frac{|N|}{|N^{(\bar{2}42)}|} = \frac{5!}{12} = \frac{120}{12} = 10$. Each single coset has four different names.

We can identify all the ten single cosets in double coset $[\bar{2}42]$ by conjugating $\bar{2}42 \sim \bar{2}32 \sim \bar{2}12 \sim \bar{2}02$ with elements of the transversal of $N^{(\bar{2}42)}$. Hence, the ten distinct single cosets are

$$\bar{2}42 \sim \bar{2}32 \sim \bar{2}12 \sim \bar{2}02$$
$$\bar{3}03 \sim \bar{3}43 \sim \bar{3}23 \sim \bar{3}13$$
$$2\bar{3}\bar{2} \sim 2\bar{4}\bar{2} \sim 2\bar{1}\bar{2} \sim 2\bar{0}\bar{2}$$
$$\bar{4}14 \sim \bar{4}04 \sim \bar{4}34 \sim \bar{4}24$$
$$4\bar{0}\bar{4} \sim 4\bar{3}\bar{4} \sim 4\bar{2}\bar{4} \sim 4\bar{1}\bar{4}$$
$$3\bar{4}\bar{3} \sim 3\bar{0}\bar{3} \sim 3\bar{1}\bar{2} \sim 3\bar{1}\bar{3}$$
$$\bar{0}30 \sim \bar{0}10 \sim \bar{0}20 \sim \bar{0}40$$
$$0\bar{1}\bar{0} \sim 0\bar{4}\bar{0} \sim 0\bar{3}\bar{0} \sim 0\bar{2}\bar{0}$$
$$\bar{1}31 \sim \bar{1}21 \sim \bar{1}01 \sim \bar{1}41$$
$$1\bar{2}\bar{1} \sim 1\bar{3}\bar{1} \sim 1\bar{0}\bar{1} \sim 1\bar{4}\bar{1}$$

The orbits of $N^{(\bar{2}42)}$ over $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{2\}$, $\{\bar{2}\}$, $(1,3,4,0)$ and $(\bar{1},\bar{3},\bar{4},\bar{0})$. Take the representatives 2, $\bar{2}$, 4 and $\bar{4}$ from each orbit, and determine which double coset $N\bar{t_2}t_4t_2t_2$, $N\bar{t_2}t_4t_2\bar{t_2}$, $N\bar{t_2}t_4t_2t_4$ and $N\bar{t_2}t_4t_2\bar{t_4}$ belong. However,

$N\bar{t_2}t_4t_2t_2 = N\bar{t_2}t_4\bar{t_2} \in [\bar{2}4\bar{2}]$

$N\bar{t_2}t_4t_2\bar{t_2} = N\bar{t_2}t_4 \in [\bar{2}4]$

$N\bar{t_2}t_4t_2t_4 \in N\bar{t_2}t_4t_2t_4N$ (New double coset denote by $[\bar{2}424]$).

$N\bar{t_2}t_4t_2\bar{t_4} \in [\bar{2}4\bar{3}]$ **(By Lemma-III)**

Therefore, six symmetric generators go back to the previous double cosets and four extend.

Next, consider new double coset $[\bar{2}424]$

Claim: $\bar{2}424 \sim \bar{2}121$

Proof. By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$ conjugating with an elements of N, we obtain the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,3,0)(\bar{1},\bar{3},\bar{0})\}} = (\bar{2}3)^{\{(1,3,0)(\bar{1},\bar{3},\bar{0})\}}$
  $\Rightarrow (2,1,4)(\bar{2},\bar{1},\bar{4})\bar{2}4 = \bar{2}1$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{0},3,\bar{2})(2,\bar{1},0,\bar{3})(4,\bar{4})\}} = (\bar{2}3)^{\{(1,\bar{0},3,\bar{2})(2,\bar{1},0,\bar{3})(4,\bar{4})\}}$
  $\Rightarrow (\bar{1},\bar{2},\bar{4})(1,2,4)1\bar{4} = 1\bar{2}$

From the relations above, we have

$\bar{2}424 = \bar{2}4\bar{2}\underline{\bar{2}4} = \bar{2}4\bar{2}(2,4,1)(\bar{2},\bar{4},\bar{1})\bar{2}1 = (2,4,1)(\bar{2},\bar{4},\bar{1})\bar{4}\underline{1\bar{4}}\bar{2}1$

$= (2,4,1)(\bar{2},\bar{4},\bar{1})\bar{4}(\bar{1},\bar{4},\bar{2})(1,4,2)1\bar{2}\bar{2}1 = \bar{2}121.$ □

Since $N^{(\bar{2}424)} \geq N^{\bar{2}424} =< (1,3,0)(\bar{1},\bar{3},\bar{0}) >$

and, $(N\bar{t_2}t_4t_2t_4)^{\{(1,3,4)(\bar{1},\bar{3},\bar{4})\}} = N\bar{t_2}t_1t_2t_1 \Rightarrow (1,3,4)(\bar{1},\bar{3},\bar{4}) \in N^{(\bar{2}424)}$

$\Rightarrow N^{(\bar{2}424)} =< (1,3,0)(\bar{1},\bar{3},\bar{0}), (1,3,4)(\bar{1},\bar{3},\bar{4}) >$

$= \{e, (1,3,0)(\bar{1},\bar{3},\bar{0}), (1,0,3)(\bar{1},\bar{0},\bar{3}), (1,4,3)(\bar{1},\bar{4},\bar{3}), (3,4,0)(\bar{3},\bar{4},\bar{0}), (1,3,4)(\bar{1},\bar{3},\bar{4}),$
$(1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4}), (1,4,0)(\bar{1},\bar{4},\bar{0}), (3,0,4)(\bar{3},\bar{0},\bar{4}), (1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0}),$
$(1,0,4)(\bar{1},\bar{0},\bar{4}), (1,4)(3,0)(\bar{1},\bar{4})(\bar{3},\bar{0})\}.$

Now, take $(N\bar{t_2}t_4t_2t_4)$ conjugates with elements of $N^{(\bar{2}424)}$, we obtain the following relations:

$$\bar{2}424 \sim \bar{2}323 \sim \bar{2}121 \sim \bar{2}020$$

Therefore, the number of single cosets in double coset $[\bar{2}424]$ are at most,

$\frac{|N|}{|N^{(242)}|} = \frac{5!}{12} = \frac{120}{12} = 10$. Each single coset has four different names.

We now can identify all the ten single cosets in double coset $[\bar{2}424]$ by conjugating $\bar{2}424 \sim \bar{2}323 \sim \bar{2}121 \sim \bar{2}020$ with elements of the transversal of $N^{(\bar{2}424)}$. Hence, the ten distinct single cosets are

$$
\begin{gathered}
\bar{2}424 \sim \bar{2}323 \sim \bar{2}121 \sim \bar{2}020 \\
\bar{3}030 \sim \bar{3}434 \sim \bar{3}232 \sim \bar{3}131 \\
2\bar{3}\bar{2}\bar{3} \sim 2\bar{4}\bar{2}\bar{4} \sim 2\bar{1}\bar{2}\bar{1} \sim 2\bar{0}\bar{2}\bar{0} \\
\bar{4}141 \sim \bar{4}040 \sim \bar{4}343 \sim \bar{4}242 \\
4\bar{0}\bar{4}\bar{0} \sim 4\bar{3}\bar{4}\bar{3} \sim 4\bar{2}\bar{4}\bar{2} \sim 4\bar{1}\bar{4}\bar{1} \\
3\bar{4}\bar{3}\bar{4} \sim 3\bar{0}\bar{3}\bar{0} \sim 3\bar{2}\bar{3}\bar{2} \sim 3\bar{1}\bar{3}\bar{1} \\
\bar{0}101 \sim \bar{0}404 \sim \bar{0}202 \sim \bar{0}303 \\
0\bar{1}\bar{0}\bar{1} \sim 0\bar{4}\bar{0}\bar{4} \sim 0\bar{3}\bar{0}\bar{3} \sim 0\bar{2}\bar{0}\bar{2} \\
\bar{1}313 \sim \bar{1}212 \sim \bar{1}010 \sim \bar{1}414 \\
1\bar{2}\bar{1}\bar{2} \sim 1\bar{3}\bar{1}\bar{3} \sim 1\bar{0}\bar{1}\bar{0} \sim 1\bar{4}\bar{1}\bar{4}
\end{gathered}
$$

The orbits of $N^{(\bar{2}424)}$ over $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{2\}$, $\{\bar{2}\}$, $(1,3,4,0)$ and $(\bar{1},\bar{3},\bar{4},\bar{0})$. We now again take representatives 2, $\bar{2}$, 4 and $\bar{4}$ from each orbit, and determine which double coset $Nt_{\bar{2}}t_4t_2t_4t_2$, $Nt_{\bar{2}}t_4t_2t_4t_{\bar{2}}$, $Nt_{\bar{2}}t_4t_2t_4t_4$ and $Nt_{\bar{2}}t_4t_2t_4t_{\bar{4}}$ belong. However,

$Nt_{\bar{2}}t_4t_2t_4t_2 \in [\bar{2}4\bar{2}]$ **(By Lemma-V)**

$Nt_{\bar{2}}t_4t_2t_4t_{\bar{2}} \in [\bar{2}4\bar{2}\bar{4}]$ **(By Lemma-IV)**

$Nt_{\bar{2}}t_4t_2t_4t_4 = Nt_{\bar{2}}t_4t_2t_{\bar{4}} \in [\bar{2}4\bar{3}]$ **(By Lemma-III)**

$Nt_{\bar{2}}t_4t_2t_4t_{\bar{4}} = Nt_{\bar{2}}t_4t_2 \in [\bar{2}42]$

Therefore, four symmetric generators go back to $[\bar{2}42]$; four go to $[\bar{2}4\bar{3}]$; one goes to $[\bar{2}4\bar{2}\bar{4}]$; and the last one goes back to $[\bar{2}4\bar{2}]$. Since all the generators go back to the existing doubles cosets and none of them extend, thus, we must have completed the double coset enumeration. The double coset enumeration shows that the index of $N \cong S_5$ in G is at most

$$|G| \leq (1+10+40+10+10+10+10+10+20+2+2+1)|N| = 126(120) = 15,120.$$

The information is summarized in the Cayley diagram below.

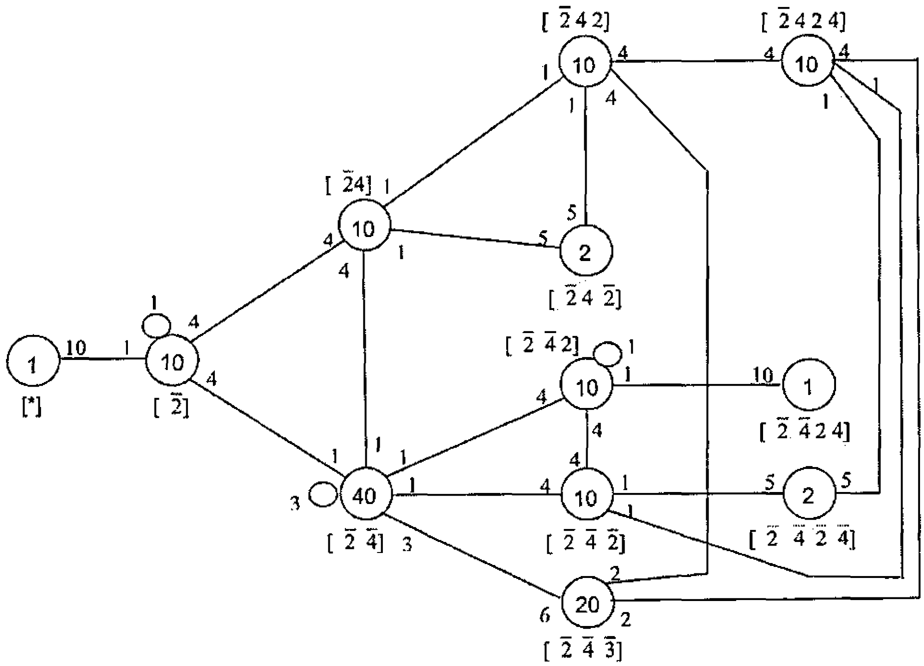

Figure 5.1: The Cayley diagram of $A_7 \times S_3$ over S_5.

The homomorphic image of G.

Define $\widehat{\alpha}: G \to S_{126}$, where (126) is a number of single cosets from the Cayley diagram above gives the action of G on the 126 cosets. From the labeling table (too large to show), we compute the actions of x, y and t_0 on the 126 cosets

$\widehat{\alpha}(x) = (2,4,8,13,5)(3,6,15,11,7)...(113,125,124,116,121)$.

$\widehat{\alpha}(y) = (2,3)(4,6)(5,11)(7,13)(8,15)(9,17)...(116,126)(117,124)(123,125)$.

$\widehat{\alpha}(t_0) = (1,2,3)(4,9,10)(5,12,14)...(112,118,125)(114,126,119)$.

We now verify that $\widehat{\alpha}$ is a homomorphism.

1. We want to show that $\widehat{\alpha}(t_0)$ has exactly 10 conjugates under conjugation by $<\widehat{\alpha}(x), \widehat{\alpha}(y)>$.

$\widehat{\alpha}(t_0)^{\widehat{\alpha}(x)} = (1,4,6)(2,27,24)(3,33,36)...(97,124,115) = \widehat{\alpha}(t_1)$.

$\widehat{\alpha}(t_1)^{\widehat{\alpha}(x)} = (1,8,15)(2,93,24)(3,33,87)...(104,125,122) = \widehat{\alpha}(t_2)$.

$\widehat{\alpha}(t_2)^{\widehat{\alpha}(x)} = (1,13,11)(2,65,24)(3,33,43)...(97,98,108) = \widehat{\alpha}(t_3)$.

$\widehat{\alpha}(t_3)^{\widehat{\alpha}(x)} = (1,5,7)(2,23,24)(3,33,35)...(95,120,123) = \widehat{\alpha}(t_4)$.

$\widehat{\alpha}(t_4)^{\widehat{\alpha}(x)} = (1,2,3)(4,9,10)(5,12,14)...(114,126,119) = \widehat{\alpha}(t_0)$.

$\widehat{\alpha}(\bar{t_0})^{\widehat{\alpha}(x)} = (1,6,4)(2,24,27)(3,36,33)...(97,115,124) = \widehat{\alpha}(\bar{t_1})$.

$\widehat{\alpha}(\bar{t_1})^{\widehat{\alpha}(x)} = (1,15,8)(2,24,93)(3,87,33)...(104,122,125) = \widehat{\alpha}(\bar{t_2})$.

$\widehat{\alpha}(\bar{t_2})^{\widehat{\alpha}(x)} = (1,11,13)(2,24,65)(3,43,33)...(97,108,98) = \widehat{\alpha}(\bar{t_3})$.

$\widehat{\alpha}(\bar{t_3})^{\widehat{\alpha}(x)} = (1,7,5)(2,24,23)(3,35,33)...(95,123,120) = \widehat{\alpha}(\bar{t_4})$.

$\widehat{\alpha}(\bar{t_4})^{\widehat{\alpha}(x)} = (1,3,2)(4,10,9)(5,14,12)...(114,119,126) = \widehat{\alpha}(\bar{t_0})$.

$\widehat{\alpha}(t_0)^{\widehat{\alpha}(y)} = (1,3,2)(4,10,9)(5,14,12)...(114,119,126) = \widehat{\alpha}(\bar{t_0})$.

$\widehat{\alpha}(\bar{t_0})^{\widehat{\alpha}(y)} = (1,2,3)(4,9,10)(5,12,14)...(114,126,119) = \widehat{\alpha}(t_0)$.

$\widehat{\alpha}(t_1)^{\widehat{\alpha}(y)} = (1,6,4)(2,24,27)(3,36,33)...(97,115,124) = \widehat{\alpha}(\bar{t_1})$.

$\widehat{\alpha}(\bar{t_1})^{\widehat{\alpha}(y)} = (1,4,6)(2,27,24)(3,33,36)...(97,124,115) = \widehat{\alpha}(t_1)$.

$\widehat{\alpha}(t_2)^{\widehat{\alpha}(y)} = (1,15,8)(2,24,93)(3,87,33)...(104,122,125) = \widehat{\alpha}(\bar{t_2})$.

$\widehat{\alpha}(\bar{t_2})^{\widehat{\alpha}(y)} = (1,8,15)(2,93,24)(3,33,87)...(104,125,122) = \widehat{\alpha}(t_2)$.

$\widehat{\alpha}(t_3)^{\widehat{\alpha}(y)} = (1,7,5)(2,24,23)(3,35,33)...(95,123,120) = \widehat{\alpha}(\bar{t_4})$.

$\widehat{\alpha}(\bar{t_4})^{\widehat{\alpha}(y)} = (1,13,11)(2,65,24)(3,33,43)...(97,98,108) = \widehat{\alpha}(t_3)$.

$\widehat{\alpha}(\bar{t_3})^{\widehat{\alpha}(y)} = (1,5,7)(2,23,24)(3,33,35)...(95,120,123) = \widehat{\alpha}(t_4)$.

$\widehat{\alpha}(t_4)^{\widehat{\alpha}(y)} = (1,11,13)(2,24,65)(3,43,33)...(97,108,98) = \widehat{\alpha}(\bar{t_3})$.

2. $\widehat{\alpha}(N)$ acts as N on $\{\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(t_2), \widehat{\alpha}(t_3), \widehat{\alpha}(t_4), \widehat{\alpha}(\bar{t_0}), \widehat{\alpha}(\bar{t_1}), \widehat{\alpha}(\bar{t_2}), \widehat{\alpha}(\bar{t_3}), \widehat{\alpha}(\bar{t_4})\}$ by conjugation, that is, if
   $x = (0,1,2,3,4)(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4})$, and
   $y = (0,\bar{0})(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})$, then
   $\widehat{\alpha}(x) = \{(\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(t_2), \widehat{\alpha}(t_3), \widehat{\alpha}(t_4))(\widehat{\alpha}(\bar{t_0}), \widehat{\alpha}(\bar{t_1}), \widehat{\alpha}(\bar{t_2}), \widehat{\alpha}(\bar{t_3}), \widehat{\alpha}(\bar{t_4}))\}$,
   $\widehat{\alpha}(y) = \{(\widehat{\alpha}(t_0), \widehat{\alpha}(\bar{t_0}))(\widehat{\alpha}(t_1), \widehat{\alpha}(\bar{t_1}))(\widehat{\alpha}(t_2), \widehat{\alpha}(\bar{t_2}))(\widehat{\alpha}(t_3), \widehat{\alpha}(\bar{t_4}))(\widehat{\alpha}(t_4), \widehat{\alpha}(\bar{t_3}))\}$.
   Then, $\widehat{\alpha}(G) = \widehat{\alpha}(3^{*5} : S_5) = \widehat{\alpha} < N, t_0 > = < \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$ is a homomorphic image of $(3^{*5} : S_5)$.
   Next, we want the homomorphic image of $G = \frac{3^{*5}:S_5}{[(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_0]^3}$.
   Now, $< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) > \leq S_{126}$ is a homomorphic image of
   $G = \frac{3^{*5}:S_5}{[(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_0]^3}$ if the additional relation
   $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{t_2}t_4 = \bar{t_2}t_3$ holds in $< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$.
   Thus, we want to show that $\widehat{\alpha}((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{t_2}t_4 = \widehat{\alpha}(\bar{t_2}t_3)$.
   On the left side of our equation
   $\widehat{\alpha}((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{t_2}t_4 = \widehat{\alpha}((2,3,4)(\bar{2},\bar{3},\bar{4}))\widehat{\alpha}(\bar{t_2})\widehat{\alpha}(t_4)$
   $= (5,8,13)(7,15,11)...(117,123,126)(1,15,8)(2,24,93)...(104,122,125)(1,5,7)$
   $(2,23,24)...(95,120,123)$
   $= (1,31,25)(3,20,29)(6,22,38)...(84,109,90)(112,115,122)$.

   On the right side of our equation
   $\widehat{\alpha}(\bar{t_2}t_3) = \widehat{\alpha}(\bar{t_0})\widehat{\alpha}(t_3)$
   $= (1,15,8)(2,24,93)...(104,122,125)(1,13,11)(2,65,24)...(97,98,108)$
   $= (1,31,25)(3,20,29)(6,22,38)...(84,109,90)(112,115,122)$.

Therefore, $\widehat{\alpha} : G \to S_{126}$ is a homomorphism map.

Now by First Isomorphism Theorem, we have

$G|ker(\widehat{\alpha}) \cong im(\widehat{\alpha})$

$\Rightarrow \frac{|G|}{|ker(\widehat{\alpha})|} = |im(\widehat{\alpha})| = | < \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) > | = 15,120$

$\Rightarrow |G| = 15,120 \times |ker(\widehat{\alpha})|$, where $(|ker(\widehat{\alpha})| \geq 1)$.

$\Rightarrow |G| \geq 15,120$.

But early we found that $|G| \leq 15,120$. Therefore, we can conclude that $|G| = 15,120$.

Since $|G| = 15,120$,

$\Rightarrow |ker(\widehat{\alpha})| = 1$.

$\Rightarrow \widehat{\alpha}$ is also an isomorphism.
$\Rightarrow G \cong \widehat{\alpha}(G) =< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$.

It is clear from the Cayley diagram of G over S_5 that every element of G can be written as πw, where π is a permutation of S_5 on ten letters, and w is a word in the t_{i_s} of the length at most four. We call this representation of elements of G the symmetric representation. We now give examples of elements of G in both symmetric and permutation representation.

Permutation Representation to Symmetric Presentation

Let $p = (1,24)(2,83)(3,62)(4,36)(5,43)(7,11)(8,87)(9,108)(12,126)(13,35)(14,30)$
$(16,100)(17,53)(18,107)(19,84)(20,123)(23,115)(25,105)(26,46)...(81,111)(92,95)$
$(93,112)(99,114)(101,122)(104,106)(116,124)$
be an element in permutation representation.
We know that $Np = Nw$
Since, $p \in Np \Rightarrow p \in Nw$.
$\Rightarrow p = nw$, where $n \in N$.
In order the represent p in the symmetric representation form, we must find $n \in N \cong S_5$.
Since, $p = nw$, where $n \in N$.
$\Rightarrow n = pw^{-1}$.
$n = pt_0\bar{t_1}$. (An element that labeling (24)).
$\Rightarrow \widehat{\alpha}(n) = p\widehat{\alpha}(t_1)\widehat{\alpha}(\bar{t_0})$
$= (2,4)(3,6)(5,13)(7,11)(9,27)(10,24)(12,38)(14,30)(16,33)(17,36)...$
$(92,95)(94,119)(97,105)(99,114)(108,110)(113,121)(116,124)(117,126)$.
From our multiplication table we have,
$n = (0,1)(3,4)(\bar{0},\bar{1})(\bar{4},\bar{3})$.
Therefore, p can be written in the symmetric representation as
$p = (0,1)(3,4)(\bar{0},\bar{1})(\bar{4},\bar{3})t_0\bar{t_1}$.

Symmetric Presentation to Permutation Representation

Let $\{(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{t_2}t_4\}$ be an element in the symmetric representation.
$\Rightarrow p = \widehat{\alpha}((2,3,4)(\bar{2},\bar{3},\bar{4}))\widehat{\alpha}(\bar{t_2})\widehat{\alpha}(t_4)$
$= (5,8,13)(7,15,11)(12,20,29)(14,21,30)...(116,125,124)(117,123,126)$

$(1,15,8)(2,24,93)(3,87,33)...(104,122,125)$
$(1,5,7)(2,23,24)(3,33,35)...(95,120,123)$
$= (1,31,25)(3,20,29)(6,22,38)(7,58,28)(8,13,47)(10,64,97)(11,103,40)(14,88,71)$
$(15,56,79)(21,77,82)(24,118,107)(26,61,32)(30,89,92)(34,59,98)(37,99,50)$
$(41,73,52)(42,74,113)(43,121,87)(44,95,55)(45,66,72)(51,96,68)(54,116,76)$
$(62,123,126)(63,114,125)(65,101,93)(70,124,120)(80,111,104)(81,106,119)$
$(84,109,90)(112,115,122)$.

Therefore, $\{(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{t_2}t_4\}$ can be written in the permutation representation as follows

$(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{t_2}t_4$
$= (1,31,25)(3,20,29)(6,22,38)(7,58,28)(8,13,47)(10,64,97)(11,103,40)(14,88,71)$
$(15,56,79)(21,77,82)(24,118,107)(26,61,32)(30,89,92)(34,59,98)(37,99,50)$
$(41,73,52)(42,74,113)(43,121,87)(44,95,55)(45,66,72)(51,96,68)(54,116,76)$
$(62,123,126)(63,114,125)(65,101,93)(70,124,120)(80,111,104)(81,106,119)$
$(84,109,90)(112,115,122)$.

5.2 Lemmas

Unless otherwise specified, we have used the relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$ and its conjugates under conjugation by$N$ in the proofs of the following lemmas.

Lemma-I: $[\bar{2}\bar{4}3] \in [\bar{2}\bar{4}]$

Proof. By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$ conjugating with an elements of N, we obtained the following relation:
$((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(2,4,3)(\bar{2},\bar{4},\bar{3})\}} = (\bar{2}3)^{\{(2,4,3)(\bar{2},\bar{4},\bar{3})\}}$
$\Rightarrow (4,2,3)(\bar{4},\bar{2},\bar{3})\bar{4}3 = \bar{4}2$
$\bar{2}\underline{\bar{4}3} = \bar{2}(4,3,2)(\bar{4},\bar{3},\bar{2})\bar{4}2 = (4,3,2)(\bar{4},\bar{3},\bar{2})\bar{4}\bar{4}2 = (4,3,2)(\bar{4},\bar{3},\bar{2})42 \in [\bar{2}\bar{4}]$. □

Lemma-II: $[\bar{2}\bar{4}2\bar{4}] \in [\bar{2}\bar{4}2]$

Proof. By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$ conjugating with an elements of N, we obtained the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{0},3,\bar{4})(2,\bar{2})(4,\bar{1},0,\bar{3})\}} = (\bar{2}3)^{\{(1,\bar{0},3,\bar{4})(2,\bar{2})(4,\bar{1},0,\bar{3})\}}$
  $\Rightarrow (2,4,1)(\bar{2},\bar{4},\bar{1})2\bar{1} = 2\bar{4}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{0},3,\bar{1},0,\bar{3})(2,\bar{4})(4,\bar{2})\}} = (\bar{2}3)^{\{(1,\bar{0},3,\bar{1},0,\bar{3})(2,\bar{4})(4,\bar{2})\}}$
  $\Rightarrow (4,1,2)(\bar{4},\bar{1},\bar{2})4\bar{2} = 4\bar{1}$

$\bar{2}\bar{4}\underline{2\bar{4}} = \bar{2}\bar{4}(2,4,1)(\bar{2},\bar{4},\bar{1})2\bar{1} = (2,4,1)(\bar{2},\bar{4},\bar{1})\bar{4}\bar{1}2\bar{1} = (2,4,1)(\bar{2},\bar{4},\bar{1})4\underline{4\bar{1}}2\bar{1} =$
$(2,4,1)(\bar{2},\bar{4},\bar{1})4(4,1,2)(\bar{4},\bar{1},\bar{2})4\bar{2}2\bar{1} = (2,1,4)(\bar{2}\bar{1}\bar{4})14\bar{1} \in [\bar{2}\bar{4}2].$ □

Lemma-III: $[\bar{2}42\bar{4}] \in [\bar{2}\bar{4}\bar{3}]$

Proof. By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$ conjugating with an elements of N, we obtained the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}} = (\bar{2}3)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}}$
  $\Rightarrow (2,4,3)(\bar{2},\bar{4},\bar{3})2\bar{3} = 2\bar{4}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(2,4,3)(\bar{2},\bar{4},\bar{3})\}} = (\bar{2}3)^{\{(2,4,3)(\bar{2},\bar{4},\bar{3})\}}$
  $\Rightarrow (4,2,3)(\bar{4},\bar{2},\bar{3})\bar{4}3 = \bar{4}2$

$\bar{2}4\underline{2\bar{4}} = \bar{2}4(2,4,3)(\bar{2},\bar{4},\bar{3})2\bar{3} = (2,4,3)(\bar{2},\bar{4},\bar{3})\underline{\bar{4}3}2\bar{3} = (2,4,3)(\bar{2},\bar{4},\bar{3})(4,3,2)(\bar{4},\bar{3},\bar{2})\bar{4}22\bar{3}$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{4}\bar{2}\bar{3} \in [\bar{2}\bar{4}\bar{3}].$ □

Lemma-IV: $[\bar{2}424\bar{2}] \in [\bar{2}\bar{4}\bar{2}\bar{4}]$

Proof. By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$ conjugating with an elements of N, we obtained the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{0},3,\bar{4})(2,\bar{2})(4,\bar{1},0,\bar{3})\}} = (\bar{2}3)^{\{(1,\bar{0},3,\bar{4})(2,\bar{2})(4,\bar{1},0,\bar{3})\}}$
  $\Rightarrow (2,4,1)(\bar{2},\bar{4},\bar{1})2\bar{1} = 2\bar{4}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,0,3,4,2)(\bar{1},\bar{0},\bar{3},\bar{4},\bar{2})\}} = (\bar{2}3)^{\{(1,0,3,4,2)(\bar{1},\bar{0},\bar{3},\bar{4},\bar{2})\}}$
  $\Rightarrow (1,4,2)(\bar{1},\bar{4},\bar{2})\bar{1}2 = \bar{1}4$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{0},3,\bar{2})(2,\bar{1},0,\bar{3})(4,\bar{4})\}} = (\bar{2}3)^{\{(1,\bar{0},3,\bar{2})(2,\bar{1},0,\bar{3})(4,\bar{4})\}}$
  $\Rightarrow (1,2,4)(\bar{1},\bar{2},\bar{4})1\bar{4} = 1\bar{2}$

$\bar{2}424\bar{2} = 2\underline{2\bar{4}}\bar{4}24\bar{2} = 2(2,4,1)(\bar{2},\bar{4},\bar{1})2\bar{1}\bar{4}24\bar{2} = (2,4,1)(\bar{2},\bar{4},\bar{1})42\bar{1}\bar{4}24\bar{2}$
$= (2,4,1)(\bar{2},\bar{4},\bar{1})421\underline{1\bar{4}}24\bar{2} = (2,4,1)(\bar{2},\bar{4},\bar{1})421(1,4,2)(\bar{1},\bar{4},\bar{2})1\bar{2}24\bar{2} = 21414\bar{2}$
$= 214\bar{1}\underline{\bar{1}4}\bar{2} = 214\bar{1}(1,4,2)(\bar{1},\bar{4},\bar{2})\bar{1}2\bar{2} = (1,4,2)(\bar{1},\bar{4},\bar{2})14\underline{2\bar{4}}\bar{1} = (1,4,2)(\bar{1},\bar{4},\bar{2})14$
$(2,4,1)(\bar{2},\bar{4},\bar{1})2\bar{1}\bar{1} = 2121 \in [\bar{2}\bar{4}\bar{2}\bar{4}]$. □

Lemma-V: $[\bar{2}4242] \in [\bar{2}\bar{4}\bar{2}]$

Proof. By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$ conjugating with an elements of N, we obtained the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3})\}} = (\bar{2}3)^{\{(1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3})\}}$
  $\Rightarrow (3,2,4)(\bar{3}\bar{2}\bar{4})\bar{3}4 = \bar{3}2$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{1})(2,\bar{4})(3,\bar{3})(4,\bar{2})(0,\bar{0})\}} = (\bar{2}3)^{\{(1,\bar{1})(2,\bar{4})(3,\bar{3})(4,\bar{2})(0,\bar{0})\}}$
  $\Rightarrow (4,3,2)(\bar{4},\bar{3},\bar{2})4\bar{2} = 4\bar{3}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{2})(2,\bar{3},4,\bar{2},3,\bar{4})(0,\bar{2})\}} = (\bar{2}3)^{\{(1,\bar{2})(2,\bar{3},4,\bar{2},3,\bar{4})(0,\bar{2})\}}$
  $\Rightarrow (3,4,2)(\bar{3},\bar{4},\bar{2})3\bar{2} = 3\bar{4}$

$\bar{2}4242 = (2,4,3)(\bar{2},\bar{4},\bar{3})\bar{2}3242 = (2,4,3)(\bar{2},\bar{4},\bar{3})\bar{2}3\underline{\bar{3}2}42$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})\bar{2}\bar{3}(3,2,4)(\bar{3}\bar{2}\bar{4})\bar{3}442 = (2,3,4)(\bar{2}\bar{3}\bar{4})\bar{4}\bar{2}\bar{3}\bar{4}2$
$= (2,3,4)(\bar{2}\bar{3}\bar{4})4\underline{4\bar{2}}\bar{3}\bar{4}2(2,3,4)(\bar{2}\bar{3}\bar{4})\bar{4}\bar{2}\bar{3}\bar{4}2 = (2,3,4)(\bar{2}\bar{3}\bar{4})4(4,2,3)(\bar{4},\bar{2},\bar{3})4\bar{3}\bar{3}\bar{4}2$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})24\underline{3\bar{4}}2 = (2,4,3)(\bar{2},\bar{4},\bar{3})24(3,4,2)(\bar{3},\bar{4},\bar{2})3\bar{2}2 = 323\bar{2}2 = 323 \in [\bar{2}\bar{4}\bar{2}]$ □

Lemma-VI: $[\bar{2}\bar{4}\bar{3}\bar{4}] \in [\bar{2}42]$

Proof. By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$ conjugating with an elements of N, we obtained the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}} = (\bar{2}3)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}}$
  $\Rightarrow (2,4,3)(\bar{2},\bar{4},\bar{3})2\bar{3} = 2\bar{4}$
- By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$

$\bar{2}\bar{4}\bar{3}\bar{4} = 2\underline{2\bar{4}}\bar{3}\bar{4} = 2(2,4,3)(\bar{2},\bar{4},\bar{3})2\bar{3}\bar{3}\bar{4} = (2,4,3)(\bar{2},\bar{4},\bar{3})423\bar{4} = (2,4,3)(\bar{2},\bar{4},\bar{3})4\bar{2}\underline{\bar{2}3}\bar{4}$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})4\bar{2}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4\bar{4} = 2\bar{3}\bar{2} \in [\bar{2}42]$. □

Lemma-VII: $[\bar{2}4\bar{3}4] \in [\bar{2}424]$

Proof. By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$ conjugating with an elements of N, we obtained the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}} = (\bar{2}3)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}}$
  $\Rightarrow (2,4,3)(\bar{2},\bar{4},\bar{3})2\bar{3} = 2\bar{4}$
- By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$

$\bar{2}\bar{4}\bar{3}4 = 2\underline{2\bar{4}}\bar{3}4 = 2(2,4,3)(\bar{2},\bar{4},\bar{3})2\bar{3}\bar{3}4 = (2,4,3)(\bar{2},\bar{4},\bar{3})4234 = (2,4,3)(\bar{2},\bar{4},\bar{3})4\bar{2}\underline{\bar{2}3}4$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})4\bar{2}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}44 = 2\bar{3}\bar{2}\bar{4} = (2,3,4)(\bar{2},\bar{3},\bar{4})2\bar{4}\bar{2}\bar{4} \in [\bar{2}424].$ □

Lemma-VIII: $[\bar{2}\bar{4}\bar{2}4] \in [\bar{2}424]$

Proof. By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$ conjugating with an elements of N, we obtained the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}} = (\bar{2}3)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}}$
  $\Rightarrow (2,4,3)(\bar{2},\bar{4},\bar{3})2\bar{3} = 2\bar{4}$
- By our relation $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4 = \bar{2}3$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(2,4,3)(\bar{2},\bar{4},\bar{3})\}} = (\bar{2}3)^{\{(2,4,3)(\bar{2},\bar{4},\bar{3})\}}$
  $\Rightarrow (4,2,3)(\bar{4},\bar{2},\bar{3})\bar{4}3 = \bar{4}2$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{2}4)^{\{(1,\bar{1})(2,\bar{4})(3,\bar{3})(4,\bar{2})(0,\bar{0})\}} = (\bar{2}3)^{\{(1,\bar{1})(2,\bar{4})(3,\bar{3})(4,\bar{2})(0,\bar{0})\}}$
  $\Rightarrow (4,3,2)(\bar{4},\bar{3},\bar{2})4\bar{2} = 4\bar{3}$

$\bar{2}\bar{4}\bar{2}4 = 2\underline{2\bar{4}}\bar{2}4 = 2(2,4,3)(\bar{2},\bar{4},\bar{3})2\bar{3}\bar{2}4 = (2,4,3)(\bar{2},\bar{4},\bar{3})42\bar{3}\underline{\bar{2}4}$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})42\bar{3}(2,4,3)(\bar{2},\bar{4},\bar{3})\bar{2}3 = (2,3,4)(\bar{2},\bar{3},\bar{4})34\bar{2}\bar{2}3$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})3423 = (2,3,4)(\bar{2},\bar{3},\bar{4})3\bar{4}\underline{\bar{4}2}3$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})3\bar{4}(4,2,3)(\bar{4},\bar{2},\bar{3})\bar{4}33 = (2,4,3)(\bar{2},\bar{4},\bar{3})\underline{4\bar{2}}\bar{4}\bar{3}$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})(4,2,3)(\bar{4},\bar{2},\bar{3})4\bar{3}\bar{4}\bar{3} = 4\bar{3}\bar{4}\bar{3} \in [\bar{2}424].$ □

Chapter 6

Construction of $3^{\cdot}A_7$

6.1 Manual double coset enumeration of G over S_4

Factoring the progenitor $3^{*3} :_m S_4$ by the relation $[(1,\bar{2})(\bar{1},2)t_1]^5$, we obtain the finite image

$$G \cong \frac{3^{*3}:_m S_4}{[(1,\bar{2})(\bar{1},2)t_1]^5=1}$$

The symmetric presentation of the progenitor $3^{*3} :_m S_4$ can be given by:

$$3^{*3} :_m S_4 \cong < x, y, t | x^4, y^2, (yx)^3, t^3, [ty], (x^2t)^2 >$$

The control group is $N = S_4 \cong < x, y | x^4 = y^2 = (yx)^3 >$. The action on the symmetric generators is given by $x \sim (0,1,\bar{0},\bar{1})(2,\bar{2})$, $y \sim (1,\bar{2})(2,\bar{1})$, and our relation is $[(1,\bar{2})(\bar{1},2)t_1]^5$.

First, we study our relation.

Let $\pi = (1,\bar{2})(2,\bar{1})$, then $(\pi t_1)^5 = 1$

$$\begin{aligned}
(\pi t_1)^5 &= 1\\
\pi t_1\pi t_1\pi t_1\pi t_1\pi t_1 &= 1\\
\pi t_1\pi t_1\pi t_1\pi\pi\pi^{-1}t_1\pi t_1 &= 1\\
\pi t_1\pi t_1\pi t_1\pi^2(t_1)^{\pi}t_1 &= 1\\
\pi t_1\pi t_1\pi\pi^2\pi^{-2}t_1\pi^2(t_1)^{\pi}t_1 &= 1\\
\pi t_1\pi\pi^3\pi^{-3}t_1\pi^3(t_1)^{\pi^2}(t_1)^{\pi}t_1 &= 1\\
\pi t_1\pi^4(t_1)^{\pi^3}(t_1)^{\pi^2}(t_1)^{\pi}t_1 &= 1\\
\pi\pi^4\pi^{-4}t_1\pi^4(t_1)^{\pi^3}(t_1)^{\pi^2}(t_1)^{\pi}t_1 &= 1\\
\pi^5(t_1)^{\pi^4}(t_1)^{\pi^3}(t_1)^{\pi^2}(t_1)^{\pi}t_1 &= 1
\end{aligned}$$

Since, $\pi = (1,\bar{2})(2,\bar{1})$
$\Rightarrow \pi^2 = (1,\bar{2})(2,\bar{1})(1,\bar{2})(2,\bar{1}) = 1.$
$\Rightarrow \pi^3 = (1,\bar{2})(2,\bar{1}).$
$\Rightarrow \pi^4 = 1.$
$\Rightarrow \pi^5 = (1,\bar{2})(2,\bar{1}).$

$$\begin{aligned}
\pi^5(t_1)^{\pi^4}(t_1)^{\pi^3}(t_1)^{\pi^2}(t_1)^{\pi}t_1 &= 1\\
\Leftrightarrow \pi^5 t_1\bar{t_2}t_1\bar{t_2}t_1 &= 1\\
\Leftrightarrow \pi^5 t_1\bar{t_2}t_1 &= \bar{t_1}t_2
\end{aligned}$$

Double coset enumeration of G over S_4.

Unless otherwise specified, proof of all the lemmas in Chapter-6 can be found on pages 86-91.

We note that,

$$NeN = \{Nen|n \in N\} = \{Nn|n \in N\} = \{N\}.$$

Let [*] represents for the double coset NeN. [*] has only one distinct right coset, namely N. Since, $N \cong S_4$ acts transitively on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$. Therefore, N has one orbit namely

$\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ in $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$. Now, we now take one representative 0 from the orbit, and find which double coset Nt_0 belongs.

Clearly, $Nt_0 \in Nt_0N = \{Nt_0^n | n \in N\} = \{Nt_0, Nt_1, Nt_2, N\bar{t_0}, N\bar{t_1}, N\bar{t_2}\}$ and let us denote Nt_0N by [0]. Now, consider the coset stabilizer $N^{(0)}$. The coset stabilizer $N^{(0)}$ is equal to the point stabilizer N^0 and is given by:

$$N^{(0)} = N^0 = \{e, (1,\bar{2})(2,\bar{1}), (1,2)(\bar{1},\bar{2}), (1,\bar{1})(2,\bar{2})\}.$$

Therefore, the number of single cosets in [0] are at most

$$\frac{|N|}{|N^{(0)}|} = \frac{4!}{4} = \frac{24}{4} = 6.$$

The orbits of $N^{(0)}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{\bar{0}\}$ and $\{1,2,\bar{1},\bar{2}\}$. Now, take one representative from each orbit, namely 0, $\bar{0}$ and 1, and find the double coset to which Nt_0t_0, $Nt_0\bar{t_0}$ and Nt_0t_1 belong? However,

$Nt_0t_0 = N\bar{t_0} \in [0]$
$Nt_0\bar{t_0} = Ne \in [*]$
$Nt_0t_1 = Nt_0t_1 \in Nt_0t_1N$ (New double coset denote by [01])

Therefore, one symmetric generator goes back to the identity double coset [*]; one stays in [0]; four extend to new double coset [01].

Next, consider the new double coset [01].

$N^{(01)} = N^{01} = \langle e \rangle$.
Therefore, the number of single cosets in the double coset [01] are at most

$$\frac{|N|}{|N^{(01)}|} = \frac{4!}{1} = \frac{24}{1} = 24.$$ Each single coset has only one name.

We can identify the 24 single cosets in the double coset [01] by conjugating Nt_0t_1 with the elements of transversal of $N^{(01)}$, we obtain the following 24 single cosets of double coset [01]:

$01, \bar{2}1, 21, \bar{0}1, 12, \bar{1}2, \bar{0}2, 02, \bar{1}0, 10, 20, \bar{2}0, 0\bar{1}, 0\bar{1}, \bar{2}\bar{1}, 2\bar{1}, 0\bar{2}, \bar{0}\bar{2}, 1\bar{2}, \bar{1}\bar{2}, 1\bar{0}, \bar{2}\bar{0}, \bar{1}\bar{0}, 2\bar{0}$.

The orbits of $N^{(01)}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1}\}$ and $\{\bar{2}\}$. Now, take one representative from each orbit, namely 0, 1, 2, $\bar{0}$, $\bar{1}$ and $\bar{2}$, and find the double cosets to which $Nt_0t_1t_0$, $Nt_0t_1t_1$, $Nt_0t_1t_2$, $Nt_0t_1\bar{t_0}$, $Nt_0t_1\bar{t_1}$ and $Nt_0t_1\bar{t_2}$ belong? However,

$Nt_0t_1t_0 = Nt_0t_1t_0 \in [01]$ **(By Lemma-1)**.
$Nt_0t_1t_1 = Nt_0\bar{t_1} \in [0\bar{1}]$.
$Nt_0t_1t_2 = Nt_0t_1t_2 \in Nt_0t_1t_2N$ (New double coset denote by [012]).
$Nt_0t_1\bar{t_0} = Nt_0t_1\bar{t_0} \in Nt_0t_1\bar{t_0}N$ (New double coset denote by $[01\bar{0}]$).
$Nt_0t_1\bar{t_1} = Nt_0 \in [0]$.
$Nt_0t_1\bar{t_2} = Nt_0t_1\bar{t_2} \in Nt_0t_1\bar{t_2}N$ (New double coset denote by $[01\bar{2}]$).

Consider the new double coset [012].

$N^{(012)} = N^{012} = < e >$.
Therefore, the number of single cosets in the double coset [012] are at most

$\frac{|N|}{|N^{(012)}|} = \frac{4!}{1} = \frac{24}{1} = 24$. Each single coset has only one name.

By conjugating $Nt_0t_1t_2$ with the elements of transversal of $N^{(012)}$, we obtain the 24 single cosets of double coset [012]:
$012, \bar{2}1\bar{0}, 210, \bar{0}1\bar{2}, 120, \bar{1}2\bar{0}, \bar{0}2\bar{1}, 021, \bar{1}0\bar{2}, 102, 201, \bar{2}0\bar{1}, \bar{0}\bar{1}2, 0\bar{1}\bar{2}, \bar{2}\bar{1}0, 2\bar{1}\bar{0}, 0\bar{2}\bar{1}, \bar{0}\bar{2}1, 1\bar{2}\bar{0}$,
$\bar{1}\bar{2}0, 1\bar{0}\bar{2}, \bar{2}\bar{0}1, \bar{1}\bar{0}2, 2\bar{0}\bar{1}$.

The orbits of $N^{(012)}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1}\}$ and $\{\bar{2}\}$. Now, take one representative from each orbit, namely 0, 1, 2, $\bar{0}$, $\bar{1}$ and $\bar{2}$, then find the double cosets to which $Nt_0t_1t_2t_0$, $Nt_0t_1t_2t_1$, $Nt_0t_1t_2t_2$, $Nt_0t_1t_2\bar{t_0}$, $Nt_0t_1t_2\bar{t_1}$ and $Nt_0t_1t_2\bar{t_2}$ belong? However,

$Nt_0t_1t_2t_0 = Nt_0t_1t_2t_0 \in Nt_0t_1t_2t_0N$ (New double coset denote by [0120]).
$Nt_0t_1t_2t_1 = Nt_0t_1t_2t_1 \in [012]$ **(By Lemma-2)**.
$Nt_0t_1t_2t_2 = Nt_0t_1\bar{t_2} \in [01\bar{2}]$.

$Nt_0t_1t_2\bar{t_0} = Nt_0t_1t_2\bar{t_0} \in Nt_0t_1t_2\bar{t_0}N$ (New double coset denote by $[012\bar{0}]$).
$Nt_0t_1t_2\bar{t_1} = Nt_0t_1t_2\bar{t_1} \in Nt_0t_1t_2\bar{t_1}N$ (New double coset denote by $[012\bar{1}]$).
$Nt_0t_1t_2\bar{t_2} = Nt_0t_1 \in [01]$.

Consider the double coset $[01\bar{0}]$
Claim: $01\bar{0} \sim 10\bar{1} \sim \bar{0}\bar{1}0$

Proof. By using our relation conjugates with elements of N, we have
1. $0\underline{1\bar{0}} = 0(\bar{1},0)(1,\bar{0})\bar{1}0\bar{1} = (\bar{1},0)(1,\bar{0})\bar{1}\bar{1}0\bar{1} = (\bar{1},0)(1,\bar{0})10\bar{1}$.

2. $\underline{01}\bar{0} = (0,1)(\bar{0},\bar{1})\bar{0}\bar{1}\bar{0}\bar{0} = (0,1)(\bar{0},\bar{1})\bar{0}\bar{1}0$. $\square$

Since $N^{(01\bar{0})} \geq N^{01\bar{0}} =< e >$ and
$N(t_0t_1\bar{t_0})^{\{(1,0)(\bar{1},\bar{0})\}} = Nt_1t_0\bar{t_1} \Rightarrow (1,0)(\bar{1},\bar{0}) \in N^{(01\bar{0})}$
$N(t_0t_1\bar{t_0})^{\{(1,\bar{1})(0,\bar{0})\}} = N\bar{t_0}\bar{t_1}t_0 \Rightarrow (1,\bar{1})(0,\bar{0}) \in N^{(01\bar{0})}$
$\Rightarrow N^{(01\bar{0})} =< (1,0)(\bar{1},\bar{0}), (1,\bar{1})(0,\bar{0}) >$
$= \{e, (1,0)(\bar{1},\bar{0}), (1,\bar{1})(0,\bar{0}), (1,\bar{0})(0,\bar{1})\}$.
Therefore, the number of single cosets in the double coset $[01\bar{0}]$ are at most

$\frac{|N|}{|N^{(01\bar{0})}|} = \frac{4!}{4} = \frac{24}{4} = 6$. Each single coset has four different names.

Now, take $(01\bar{0})$ conjugates with the elements of $N^{(01\bar{0})}$, we obtain the following relations:

$$01\bar{0} \sim 10\bar{1} \sim \bar{0}\bar{1}0 \sim \bar{1}\bar{0}1$$

By conjugating $(01\bar{0} \sim 10\bar{1} \sim \bar{0}\bar{1}0 \sim \bar{1}\bar{0}1)$ with the elements of transversal of $N^{(01\bar{0})}$, we obtain the six single cosets of double coset $[01\bar{0}]$:

$$\begin{gathered}
01\bar{0} \sim 10\bar{1} \sim \bar{0}\bar{1}0 \sim \bar{1}\bar{0}1 \\
\bar{0}10 \sim 1\bar{0}\bar{1} \sim 0\bar{1}\bar{0} \sim \bar{1}01 \\
0\bar{2}\bar{0} \sim \bar{2}02 \sim \bar{0}20 \sim 2\bar{0}\bar{2} \\
\bar{0}\bar{2}0 \sim \bar{2}\bar{0}2 \sim 02\bar{0} \sim 20\bar{2} \\
12\bar{1} \sim 21\bar{2} \sim \bar{1}\bar{2}1 \sim \bar{2}\bar{1}2 \\
\bar{1}21 \sim 2\bar{1}\bar{2} \sim 1\bar{2}\bar{1} \sim \bar{2}12
\end{gathered}$$

The orbits of $N^{(01\bar{0})}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{2\}$, $\{\bar{2}\}$ and $\{1,0,\bar{1},\bar{0}\}$. Now, take one representative from each orbit, namely 2, $\bar{2}$ and 0, and find the double coset to which $Nt_0t_1\bar{t_0}t_2$, $Nt_0t_1\bar{t_0}\bar{t_2}$ and $Nt_0t_1\bar{t_0}t_0$ belong? However,

$Nt_0t_1\bar{t_0}t_2 = Nt_0t_1\bar{t_0}t_2 \in [012\bar{1}]$ **(By Lemma-4).**
$Nt_0t_1\bar{t_0}\bar{t_2} = Nt_0t_1\bar{t_0}\bar{t_2} \in Nt_0t_1\bar{t_0}\bar{t_2}N$ (New double coset denote by $[01\bar{0}\bar{2}]$).
$Nt_0t_1\bar{t_0}t_0 = Nt_0t_1 \in [01]$.

Consider the double coset $[01\bar{2}]$.

$N^{(01\bar{2})} = N^{01\bar{2}} =< e >$.
Therefore, the number of single cosets in the double coset $[01\bar{2}]$ are at most

$$\frac{|N|}{|N^{(01\bar{2})}|}=\frac{4!}{1}=\frac{24}{1}=24.$$ Each single coset has only one name.

By conjugating $(01\bar{2})$ with the elements of transversal of $N^{(01\bar{2})}$, we obtain the 24 single cosets of double coset $[01\bar{2}]$:
$01\bar{2}, \bar{2}10, 21\bar{0}, \bar{0}12, 12\bar{0}, \bar{1}20, \bar{0}21, 02\bar{1}, \bar{1}02, 10\bar{2}, 20\bar{1}, \bar{2}01, \bar{0}\bar{1}\bar{2}, 0\bar{1}2, \bar{2}\bar{1}\bar{0}, 2\bar{1}0, 0\bar{2}1, \bar{0}\bar{2}\bar{1}, 1\bar{2}0,$
$\bar{1}\bar{2}\bar{0}, 1\bar{0}2, \bar{2}\bar{0}\bar{1}, \bar{1}\bar{0}\bar{2}, 2\bar{0}1.$

The orbits of $N^{(01\bar{2})}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1}\}$ and $\{\bar{2}\}$. Now, take one representative from each orbit, namely 0, 1, 2, $\bar{0}$, $\bar{1}$ and $\bar{2}$, and find the double cosets to which $Nt_0t_1\bar{t_2}t_0$, $Nt_0t_1\bar{t_2}t_1$, $Nt_0t_1\bar{t_2}t_2$, $Nt_0t_1\bar{t_2}\bar{t_0}$, $Nt_0t_1\bar{t_2}\bar{t_1}$ and $Nt_0t_1\bar{t_2}\bar{t_2}$ belong? However,

$Nt_0t_1\bar{t_2}t_0 = Nt_0t_1\bar{t_2}t_0 \in Nt_0t_1\bar{t_2}t_0N$ (New double coset denote by $[01\bar{2}0]$).
$Nt_0t_1\bar{t_2}t_1 = Nt_0t_1\bar{t_2}t_1 \in [01\bar{2}]$.
$Nt_0t_1\bar{t_2}t_2 = Nt_0t_1 \in [01]$.
$Nt_0t_1\bar{t_2}\bar{t_0} = Nt_0t_1\bar{t_2}\bar{t_0} \in [012\bar{0}]$ **(By Lemma-3).**
$Nt_0t_1\bar{t_2}\bar{t_1} = Nt_0t_1\bar{t_2}\bar{t_1} \in Nt_0t_1\bar{t_2}\bar{t_1}N$ (New double coset denote by $[01\bar{2}\bar{1}]$).
$Nt_0t_1\bar{t_2}\bar{t_2} = Nt_0t_1t_2 \in [012]$.

Consider the new double coset $[0120]$.

$N^{(0120)} = N^{0120} =< e >$.

Therefore, the number of single cosets in the double coset [0120] are at most

$\frac{|N|}{|N^{(0120)}|}=\frac{4!}{1} = \frac{24}{1} = 24$. Each single coset has only one name.

By conjugating (0120) with the elements of transversal of $N^{(0120)}$, we obtain the 24 single cosets of double coset [0120]:

$0120, \bar{2}1\bar{0}\bar{2}, 2102, \bar{0}1\bar{2}\bar{0}, 1201, \bar{1}\bar{2}0\bar{1}, \bar{0}\bar{2}\bar{1}\bar{0}, 0210, \bar{1}0\bar{2}\bar{1}, 1021, 2012, \bar{2}0\bar{1}\bar{2}, \bar{0}\bar{1}\bar{2}0, 0\bar{1}\bar{2}0, \bar{2}\bar{1}0\bar{2},$
$2\bar{1}\bar{0}2, 0\bar{2}\bar{1}0, \bar{0}\bar{2}1\bar{0}, 1\bar{2}\bar{0}1, \bar{1}20\bar{1}, 1\bar{0}\bar{2}1, \bar{2}\bar{0}1\bar{2}, \bar{1}\bar{0}2\bar{1}, 2\bar{0}\bar{1}2.$

The orbits of $N^{(0120)}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1}\}$ and $\{\bar{2}\}$. Now, take one representative from each orbit, namely 0, 1, 2, $\bar{0}$, $\bar{1}$ and $\bar{2}$, then find the double cosets to which $Nt_0t_1t_2t_0t_0$, $Nt_0t_1t_2t_0t_1$, $Nt_0t_1t_2t_0t_2$, $Nt_0t_1t_2t_0t_{\bar{0}}$, $Nt_0t_1t_2t_0t_{\bar{1}}$ and $Nt_0t_1t_2t_0t_{\bar{2}}$ belong? However,

$Nt_0t_1t_2t_0t_0 = Nt_0t_1t_2t_{\bar{0}} \in [012\bar{0}]$.

$Nt_0t_1t_2t_0t_1 = Nt_0t_1t_2t_0t_1 \in Nt_0t_1t_2t_0t_1N$ (New double coset denote by [01201]).

$Nt_0t_1t_2t_0t_2 = Nt_0t_1t_2t_0t_2 \in [01\bar{0}\bar{2}]$ **(By Lemma-4)**.

$Nt_0t_1t_2t_0t_{\bar{0}} = Nt_0t_1t_2 \in [012]$.

$Nt_0t_1t_2t_0t_{\bar{1}} = Nt_0t_1t_2t_0t_{\bar{1}} \in Nt_0t_1t_2t_0t_{\bar{1}}N$ (New double coset denote by $[0120\bar{1}]$).

$Nt_0t_1t_2t_0t_{\bar{2}} = Nt_0t_1t_2t_0t_{\bar{2}} \in [0120]$ **(By Lemma-5)**.

Consider the new double coset $[012\bar{0}]$.

$N^{(012\bar{0})} = N^{012\bar{0}} =< e >$.

Therefore, the number of single cosets in the double coset $[012\bar{0}]$ are at most

$\frac{|N|}{|N^{(012\bar{0})}|}=\frac{4!}{1} = \frac{24}{1} = 24$. Each single coset has only one name.

By conjugating $(012\bar{0})$ with the elements of transversal of $N^{(012\bar{0})}$, we obtain the 24 single cosets of double coset $[012\bar{0}]$:

$012\bar{0}, \bar{2}1\bar{0}2, 210\bar{2}, \bar{0}1\bar{2}0, 120\bar{1}, \bar{1}\bar{2}01, \bar{0}\bar{2}\bar{1}0, 021\bar{0}, \bar{1}0\bar{2}1, 102\bar{1}, 201\bar{2}, \bar{2}0\bar{1}2, \bar{0}\bar{1}20, 0\bar{1}\bar{2}\bar{0}, \bar{2}\bar{1}02,$
$2\bar{1}\bar{0}\bar{2}, 0\bar{2}\bar{1}\bar{0}, \bar{0}\bar{2}10, 1\bar{2}\bar{0}\bar{1}, \bar{1}201, 1\bar{0}\bar{2}\bar{1}, \bar{2}\bar{0}12, \bar{1}\bar{0}21, 2\bar{0}\bar{1}\bar{2}.$

The orbits of $N^{(012\bar{0})}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1}\}$ and $\{\bar{2}\}$. Now, take one representative from each orbit, namely 0, 1, 2, $\bar{0}$, $\bar{1}$ and $\bar{2}$, and find the double cosets to which $Nt_0t_1t_2\bar{t_0}t_0$, $Nt_0t_1t_2\bar{t_0}t_1$, $Nt_0t_1t_2\bar{t_0}t_2$, $Nt_0t_1t_2\bar{t_0}\bar{t_0}$, $Nt_0t_1t_2\bar{t_0}\bar{t_1}$ and $Nt_0t_1t_2\bar{t_0}\bar{t_2}$ belong? However,

$Nt_0t_1t_2\bar{t_0}t_0 = Nt_0t_1t_2 \in [012]$.
$Nt_0t_1t_2\bar{t_0}t_1 = Nt_0t_1t_2\bar{t_0}t_1 \in Nt_0t_1t_2\bar{t_0}t_1N$ (New double coset denote by $[012\bar{0}2]$.
$Nt_0t_1t_2\bar{t_0}t_2 = Nt_0t_1t_2\bar{t_0}t_2 \in [01\bar{2}]$ **(By Lemma-6).**
$Nt_0t_1t_2\bar{t_0}\bar{t_0} = Nt_0t_1t_2t_0 \in [0120]$.
$Nt_0t_1t_2\bar{t_0}\bar{t_1} = Nt_0t_1t_2\bar{t_0}\bar{t_1} \in [0120\bar{1}]$ **(By Lemma-7).**
$Nt_0t_1t_2\bar{t_0}\bar{t_2} = Nt_0t_1t_2\bar{t_0}\bar{t_2} \in [01\bar{2}0]$ **(By Lemma-8).**

Consider the new double coset $[012\bar{1}]$.
Claim: $012\bar{1} \sim 0\bar{2}\bar{1}2 \sim 021\bar{2}$

Proof. By using our relation conjugates with elements of N, we have
1. $01\underline{2\bar{1}} = 01(\bar{2},1)(2,\bar{1})\bar{2}1\bar{2} = (\bar{2},1)(2,\bar{1})0\bar{2}\bar{2}1\bar{2} = (\bar{2},1)(2,\bar{1})021\bar{2} = (\bar{2},1)(2,\bar{1})0\underline{212}2$
$= (\bar{2},1)(2,\bar{1})0(1,2)(\bar{1},\bar{2})\bar{2}\bar{1}2 = (1,\bar{1})(2,\bar{2})0\bar{2}\bar{1}2$.

2. $01\underline{2\bar{1}} = 01(\bar{2},1)(2,\bar{1})\bar{2}1\bar{2} = (\bar{2},1)(2,\bar{1})0\bar{2}\bar{2}1\bar{2} = (\bar{2},1)(2,\bar{1})021\bar{2}$. □

Since $N^{(012\bar{1})} \geq N^{012\bar{1}} =< e >$ and
$N(t_0t_1t_2\bar{t_1})^{\{(1,\bar{2})(\bar{1},2)\}} = Nt_0\bar{t_2}\bar{t_1}t_2 \Rightarrow (1,\bar{2})(\bar{1},2) \in N^{(012\bar{1})}$
$N(t_0t_1t_2\bar{t_1})^{\{(1,2)(\bar{1},\bar{2})\}} = Nt_0t_2t_1\bar{t_2} \Rightarrow (1,2)(\bar{1},\bar{2}) \in N^{(012\bar{1})}$
$\Rightarrow N^{(012\bar{1})} =< (1,\bar{2})(\bar{1},2), (1,2)(\bar{1},\bar{2}) >$
$= \{e, (1,\bar{2})(2,\bar{1}), (1,2)(\bar{1},\bar{2}), (1,\bar{1})(2,\bar{2})\}$.
Therefore, the number of single cosets in the double coset $[012\bar{1}]$ are at most

$\frac{|N|}{|N^{(012\bar{1})}|} = \frac{4!}{4} = \frac{24}{4} = 6$. Each single coset has four different names.

Now, take $(012\bar{1})$ conjugates with the elements of $N^{(012\bar{1})}$, we obtain the following relations:

$$012\bar{1} \sim 0\bar{2}\bar{1}2 \sim 021\bar{2} \sim 0\bar{1}\bar{2}1$$

By conjugating $(012\bar{1} \sim 0\bar{2}\bar{1}2 \sim 021\bar{2} \sim 0\bar{1}\bar{2}1)$ with the elements of transversal of $N^{(012\bar{1})}$, we obtain the six single cosets of double coset $[012\bar{1}]$:

$$0 1 2 \bar{1} \sim 0 \bar{2} \bar{1} 2 \sim 0 2 1 \bar{2} \sim 0 \bar{1} \bar{2} 1$$
$$1 \bar{0} \bar{2} 0 \sim 1 2 0 \bar{2} \sim 1 \bar{2} \bar{0} 2 \sim 1 0 2 \bar{0}$$
$$\bar{0} \bar{1} 2 1 \sim \bar{0} \bar{2} 1 2 \sim \bar{0} 2 \bar{1} \bar{2} \sim \bar{0} 1 \bar{2} \bar{1}$$
$$\bar{2} 0 1 0 \sim \bar{2} \bar{1} 0 1 \sim \bar{2} 1 \bar{0} \bar{1} \sim \bar{2} 0 \bar{1} \bar{0}$$
$$\bar{1} 0 \bar{2} \bar{0} \sim \bar{1} 2 \bar{0} \bar{2} \sim \bar{1} \bar{2} 0 2 \sim \bar{1} \bar{0} 2 0$$
$$2 0 1 \bar{0} \sim 2 \bar{1} \bar{0} 1 \sim 2 1 0 \bar{1} \sim 2 \bar{0} \bar{1} 0$$

The orbits of $N^{(012\bar{1})}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{\bar{0}\}$ and $\{1,2,\bar{1},\bar{2}\}$. Now, take one representative from each orbit, namely 0, $\bar{0}$ and 1, then find the double coset to which $Nt_0t_1t_2\bar{t_1}t_0$, $Nt_0t_1t_2\bar{t_1}\bar{t_0}$ and $Nt_0t_1t_2\bar{t_1}t_1$ belong? However,

$Nt_0t_1t_2\bar{t_1}t_0 = Nt_0t_1t_2\bar{t_1}t_0 \in Nt_0t_1t_2\bar{t_1}t_0N$ (New double coset denote by $[012\bar{1}0]$).
$Nt_0t_1t_2\bar{t_1}\bar{t_0} = Nt_0t_1t_2\bar{t_1}\bar{t_0} \in Nt_0t_1t_2\bar{t_1}\bar{t_0}N$ (New double coset denote by $[012\bar{1}\bar{0}]$).
$Nt_0t_1t_2\bar{t_1}t_1 = Nt_0t_1t_2 \in [012]$.

Consider the double coset $[01\bar{0}2]$
Claim: $01\bar{0}2 \sim 10\bar{1}2 \sim \bar{0}\bar{1}02$

Proof. By using our relation conjugates with elements of N, we have
1. $0\underline{1\bar{0}}2 = 0(\bar{1},0)(1,\bar{0})\bar{1}0\bar{1}2 = (\bar{1},0)(1,\bar{0})\bar{1}\bar{1}0\bar{1}2 = (\bar{1},0)(1,\bar{0})10\bar{1}2.$

2. $\underline{01}\bar{0}2 = (0,1)(\bar{0},\bar{1})\bar{0}\bar{1}\bar{0}\bar{0}2 = (0,1)(\bar{0},\bar{1})\bar{0}\bar{1}02.$ ☐

Since $N^{(01\bar{0}2)} \geq N^{01\bar{0}2} =< e >$ and
$N(t_0t_1\bar{t_0}t_2)^{\{(1,0)(\bar{1},\bar{0})\}} = Nt_1t_0\bar{t_1}t_2 \Rightarrow (1,0)(\bar{1},\bar{0}) \in N^{(01\bar{0}2)}$
$N(t_0t_1\bar{t_0}t_2)^{\{(1,\bar{1})(0,\bar{0})\}} = N\bar{t_0}\bar{t_1}t_0t_2 \Rightarrow (1,\bar{1})(0,\bar{0}) \in N^{(01\bar{0}2)}$
$\Rightarrow N^{(01\bar{0}2)} =< (1,0)(\bar{1},\bar{0}), (1,\bar{1})(0,\bar{0}) >$
$= \{e, (1,0)(\bar{1},\bar{0}), (1,\bar{1})(0,\bar{0}), (1,\bar{0})(0,\bar{1})\}.$
Therefore, the number of single cosets in the double coset $[01\bar{0}2]$ are at most

$\frac{|N|}{|N^{(01\bar{0}2)}|} = \frac{4!}{4} = \frac{24}{4} = 6$. Each single coset has four different names.

Now, take $(01\bar{0}2)$ conjugates with the elements of $N^{(01\bar{0})2}$, we obtain the following relations:

$$01\bar{0}2 \sim 10\bar{1}2 \sim \bar{0}\bar{1}02 \sim \bar{1}\bar{0}12$$

By conjugating ($01\bar{0}2 \sim 10\bar{1}2 \sim \bar{0}\bar{1}02 \sim \bar{1}\bar{0}12$) with the elements of transversal of $N^{(01\bar{0}2)}$, we obtain the six single cosets of double coset $[01\bar{0}2]$:

$$\begin{array}{c} 01\bar{0}2 \sim 10\bar{1}2 \sim \bar{0}\bar{1}02 \sim \bar{1}\bar{0}12 \\ \bar{0}10\bar{2} \sim 1\bar{0}\bar{1}\bar{2} \sim 0\bar{1}\bar{0}\bar{2} \sim \bar{1}01\bar{2} \\ 0\bar{2}\bar{0}\bar{1} \sim \bar{2}02\bar{1} \sim \bar{0}20\bar{1} \sim 2\bar{0}\bar{2}\bar{1} \\ \bar{0}\bar{2}01 \sim \bar{2}\bar{0}21 \sim 02\bar{0}1 \sim 20\bar{2}1 \\ 12\bar{1}0 \sim 21\bar{2}0 \sim \bar{1}\bar{2}10 \sim \bar{2}\bar{1}20 \\ \bar{1}21\bar{0} \sim 2\bar{1}\bar{2}\bar{0} \sim 1\bar{2}\bar{1}\bar{0} \sim \bar{2}12\bar{0} \end{array}$$

The orbits of $N^{(01\bar{0}2)}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{2\}$, $\{\bar{2}\}$ and $\{1,0,\bar{1},\bar{0}\}$. Now, take one representative from each orbit, namely 2, $\bar{2}$ and 0, then find the double coset to which $Nt_0t_1\bar{t_0}t_2t_2$, $Nt_0t_1\bar{t_0}t_2\bar{t_2}$ and $Nt_0t_1\bar{t_0}t_2t_0$ belong? However,

$Nt_0t_1\bar{t_0}t_2t_2 = Nt_0t_1\bar{t_0}\bar{t_2} \in [01\bar{0}\bar{2}]$.
$Nt_0t_1\bar{t_0}t_2\bar{t_2} = Nt_0t_1\bar{t_0} \in [01\bar{0}]$.
$Nt_0t_1\bar{t_0}t_2t_0 = Nt_0t_1\bar{t_0}t_2t_0 \in [01\bar{2}0]$ **(By Lemma-9).**

Consider the double coset $[01\bar{0}\bar{2}]$
Claim: $01\bar{0}\bar{2} \sim 10\bar{1}\bar{2} \sim \bar{0}\bar{1}0\bar{2}$

Proof. By using our relation conjugates with elements of N, we have
1. $0\underline{1\bar{0}}\bar{2} = 0(\bar{1},0)(1,\bar{0})\bar{1}0\bar{1}\bar{2} = (\bar{1},0)(1,\bar{0})\bar{1}\bar{1}0\bar{1}\bar{2} = (\bar{1},0)(1,\bar{0})10\bar{1}\bar{2}$.

2. $0\underline{1\bar{0}}\bar{2} = 0(\bar{1},0)(1,\bar{0})\bar{1}0\bar{1}\bar{2} = (\bar{1},0)(1,\bar{0})\bar{1}\bar{1}0\bar{1}\bar{2} = (\bar{1},0)(1,\bar{0})\underline{10}\bar{1}\bar{2}$
$= (\bar{1},0)(1,\bar{0})(1,0)(\bar{1},\bar{0})\bar{1}\bar{0}\bar{1}\bar{1}\bar{2} = (1,\bar{1})(0,\bar{0})\bar{1}\bar{0}1\bar{2}$. ∎

Since $N^{(01\bar{0}\bar{2})} \geq N^{01\bar{0}\bar{2}} = < e >$ and
$N(t_0t_1\bar{t_0}\bar{t_2})^{\{(1,0)(\bar{1},\bar{0})\}} = Nt_1t_0\bar{t_1}\bar{t_2} \Rightarrow (1,0)(\bar{1},\bar{0}) \in N^{(01\bar{0}\bar{2})}$
$N(t_0t_1\bar{t_0}\bar{t_2})^{\{(1,\bar{0})(0,\bar{1})\}} = N\bar{t_1}\bar{t_0}t_1\bar{t_2} \Rightarrow (1,\bar{0})(0,\bar{1}) \in N^{(01\bar{0}\bar{2})}$
$\Rightarrow N^{(01\bar{0}\bar{2})} = < (1,0)(\bar{1},\bar{0}), (1,\bar{0})(0,\bar{1}) >$
$= \{e, (1,0)(\bar{1},\bar{0}), (1,\bar{1})(0,\bar{0}), (1,\bar{0})(0,\bar{1})\}$.
Therefore, the number of single cosets in the double coset $[01\bar{0}\bar{2}]$ are at most

$\frac{|N|}{|N^{(01\bar{0}\bar{2})}|} = \frac{4!}{4} = \frac{24}{4} = 6$. Each has four different names.

Now, take $(01\bar{0}\bar{2})$ conjugates with the elements of $N^{(01\bar{0}\bar{2})}$, we obtain the following relations:

$$01\bar{0}\bar{2} \sim 10\bar{1}\bar{2} \sim \bar{0}\bar{1}0\bar{2} \sim \bar{1}\bar{0}1\bar{2}$$

By conjugating $(01\bar{0}\bar{2} \sim 10\bar{1}\bar{2} \sim \bar{0}\bar{1}0\bar{2} \sim \bar{1}\bar{0}1\bar{2})$ with the elements of transversal of $N^{(01\bar{0}\bar{2})}$, we obtain the six single cosets of double coset $[01\bar{0}\bar{2}]$:

$$01\bar{0}\bar{2} \sim 10\bar{1}\bar{2} \sim \bar{0}\bar{1}0\bar{2} \sim \bar{1}\bar{0}1\bar{2}$$
$$\bar{0}102 \sim 1\bar{0}\bar{1}2 \sim 0\bar{1}\bar{0}2 \sim \bar{1}012$$
$$0\bar{2}\bar{0}1 \sim \bar{2}021 \sim \bar{0}201 \sim 2\bar{0}\bar{2}1$$
$$\bar{0}\bar{2}0\bar{1} \sim \bar{2}\bar{0}2\bar{1} \sim 02\bar{0}\bar{1} \sim 20\bar{2}\bar{1}$$
$$12\bar{1}\bar{0} \sim 21\bar{2}\bar{0} \sim \bar{1}\bar{2}1\bar{0} \sim \bar{2}\bar{1}2\bar{0}$$
$$\bar{1}210 \sim 2\bar{1}\bar{2}0 \sim 1\bar{2}\bar{1}0 \sim \bar{2}120$$

The orbits of $N^{(01\bar{0}\bar{2})}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{2\}$, $\{\bar{2}\}$ and $\{1,0,\bar{1},\bar{0}\}$. Now, take one representative from each orbit, namely 2, $\bar{2}$ and 0, then find the double coset to which $Nt_0t_1\bar{t_0}\bar{t_2}t_2$, $Nt_0t_1\bar{t_0}\bar{t_2}\bar{t_2}$ and $Nt_0t_1\bar{t_0}\bar{t_2}t_0$ belong? However,

$Nt_0t_1\bar{t_0}\bar{t_2}t_2 = Nt_0t_1\bar{t_0} \in [01\bar{0}]$.
$Nt_0t_1\bar{t_0}\bar{t_2}\bar{t_2} = Nt_0t_1\bar{t_0}t_2 \in [01\bar{0}2]$.
$Nt_0t_1\bar{t_0}\bar{t_2}t_0 = Nt_0t_1\bar{t_0}\bar{t_2}t_0 \in [0120]$ **(By Lemma-10).**

Consider the double coset $[01\bar{2}0]$.

$N^{(01\bar{2}0)} = N^{01\bar{2}0} =< e >$.
Therefore, the number of single cosets in the double coset $[01\bar{2}0]$ are at most

$\frac{|N|}{|N^{(01\bar{2}0)}|}=\frac{4!}{1} = \frac{24}{1} = 24$. Each single coset has only one name.

By conjugating $(01\bar{2}0)$ with the elements of transversal of $N^{(01\bar{2}0)}$, we obtain the 24 single cosets of double coset $[01\bar{2}0]$:
$01\bar{2}0, \bar{2}10\bar{2}, 21\bar{0}2, \bar{0}12\bar{0}, 12\bar{0}1, \bar{1}20\bar{1}, \bar{0}2\bar{1}\bar{0}, 02\bar{1}0, \bar{1}02\bar{1}, 10\bar{2}1, 20\bar{1}2, \bar{2}01\bar{2}, \bar{0}\bar{1}\bar{2}\bar{0}, 0\bar{1}20, \bar{2}\bar{1}0\bar{2},$
$2\bar{1}02, 0\bar{2}10, \bar{0}\bar{2}\bar{1}\bar{0}, 1\bar{2}01, \bar{1}\bar{2}\bar{0}\bar{1}, 1\bar{0}21, \bar{2}\bar{0}\bar{1}\bar{2}, \bar{1}\bar{0}\bar{2}\bar{1}, 2\bar{0}12.$

The orbits of $N^{(01\bar{2}0)}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1}\}$ and $\{\bar{2}\}$. Now, take one representative from each orbit, namely 0, 1, 2, $\bar{0}$, $\bar{1}$ and $\bar{2}$, then find the double cosets to which $Nt_0t_1\bar{t_2}t_0t_0$, $Nt_0t_1\bar{t_2}t_0t_1$, $Nt_0t_1\bar{t_2}t_0t_2$, $Nt_0t_1\bar{t_2}t_0\bar{t_0}$, $Nt_0t_1\bar{t_2}t_0\bar{t_1}$ and $Nt_0t_1\bar{t_2}t_0\bar{t_2}$ belong? However,

$Nt_0t_1\bar{t_2}t_0t_0 = Nt_0t_1\bar{t_2}\bar{t_0} \in [012\bar{0}]$ **(By Lemma-3).**
$Nt_0t_1\bar{t_2}t_0t_1 = Nt_0t_1\bar{t_2}t_0t_1 \in Nt_0t_1\bar{t_2}t_0t_1N$ (New double coset denoted by $[01\bar{2}01]$).
$Nt_0t_1\bar{t_2}t_0t_2 = Nt_0t_1\bar{t_2}t_0t_2 \in [01\bar{2}0]$ **(By Lemma-11).**
$Nt_0t_1\bar{t_2}t_0\bar{t_0} = Nt_0t_1\bar{t_2} \in [01\bar{2}]$.
$Nt_0t_1\bar{t_2}t_0\bar{t_1} = Nt_0t_1\bar{t_2}t_0\bar{t_1} \in [0120\bar{1}]$ **(By Lemma-12).**
$Nt_0t_1\bar{t_2}t_0\bar{t_2} = Nt_0t_1\bar{t_2}t_0\bar{t_2} \in [01\bar{0}2]$ **(By Lemma-13).**

Consider the double coset $[01\bar{2}\bar{1}]$
Claim: $01\bar{2}\bar{1} \sim 02\bar{1}\bar{2} \sim 0\bar{1}21$

Proof. By using our relation conjugates with elements of N, we have
1. $01\underline{\bar{2}\bar{1}} = 01(1,2)(\bar{1},\bar{2})212 = (1,2)(\bar{1},\bar{2})02212 = (1,2)(\bar{1},\bar{2})0\underline{21}2$
$= (1,2)(\bar{1},\bar{2})0(2,\bar{1})(\bar{2},1)2\bar{1}22 = (1,\bar{1})(\bar{2},2)02\bar{1}\bar{2}$.

2. $01\bar{2}\bar{1} = 01\underline{\bar{2}1}1 = 01(2,\bar{1})(\bar{2},1)2\bar{1}21 = (2,\bar{1})(\bar{2},1)0\bar{2}2\bar{1}21 = (2,\bar{1})(\bar{2},1)0\bar{1}21$. ☐

Since $N^{(01\bar{2}\bar{1})} \geq N^{01\bar{2}\bar{1}} =< e >$ and
$N(t_0t_1\bar{t_2}\bar{t_1})^{\{(1,2)(\bar{1},\bar{2})\}} = Nt_0t_2\bar{t_1}\bar{t_2} \Rightarrow (1,2)(\bar{1},\bar{2}) \in N^{(01\bar{2}\bar{1})}$
$N(t_0t_1\bar{t_2}\bar{t_1})^{\{(1,\bar{1})(2,\bar{2})\}} = Nt_0\bar{t_1}t_2t_1 \Rightarrow (1,\bar{1})(2,\bar{2}) \in N^{(01\bar{2}\bar{1})}$
$\Rightarrow N^{(01\bar{2}\bar{1})} =< (1,2)(\bar{1},\bar{2}), (1,\bar{1})(2,\bar{2}) >$
$= \{e, (1,2)(\bar{1},\bar{2}), (1,\bar{1})(2,\bar{2}), (1,\bar{2})(2,\bar{1})\}$
Therefore, the number of single cosets in double coset $[01\bar{2}\bar{1}]$ are at most

$\frac{|N|}{|N^{(01\bar{2}\bar{1})}|} = \frac{4!}{4} = \frac{24}{4} = 6$. Each single coset has four different names.

Now, take $(01\bar{2}\bar{1})$ conjugates with the elements of $N^{(01\bar{2}\bar{1})}$, we obtain the following relations:

$$01\bar{2}\bar{1} \sim 02\bar{1}\bar{2} \sim 0\bar{1}21 \sim 0\bar{2}12$$

By conjugating $(01\bar{2}\bar{1} \sim 02\bar{1}\bar{2} \sim 0\bar{1}21 \sim 0\bar{2}12)$ with the elements of transversal of $N^{(01\bar{2}\bar{1})}$, we obtain the following six single cosets of double coset $[01\bar{2}\bar{1}]$:

$$
\begin{gathered}
01\bar{2}\bar{1} \sim 02\bar{1}\bar{2} \sim 0\bar{1}21 \sim 0\bar{2}12 \\
1\bar{0}20 \sim 1\bar{2}02 \sim 10\bar{2}\bar{0} \sim 12\bar{0}\bar{2} \\
\bar{0}\bar{1}\bar{2}1 \sim \bar{0}21\bar{2} \sim \bar{0}12\bar{1} \sim \bar{0}\bar{2}\bar{1}2 \\
\bar{2}\bar{0}\bar{1}0 \sim \bar{2}10\bar{1} \sim \bar{2}01\bar{0} \sim \bar{2}\bar{1}\bar{0}1 \\
\bar{1}02\bar{0} \sim \bar{1}\bar{2}\bar{0}2 \sim \bar{1}\bar{0}\bar{2}0 \sim \bar{1}20\bar{2} \\
20\bar{1}\bar{0} \sim 21\bar{0}\bar{1} \sim 2\bar{0}10 \sim 2\bar{1}01
\end{gathered}
$$

The orbits of $N^{(01\bar{2}\bar{1})}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{\bar{0}\}$ and $\{1,2,\bar{1},\bar{2}\}$. Now, take one representative from each orbit, namely 0, $\bar{0}$ and 1, then find the double coset to which $Nt_0t_1\bar{t_2}\bar{t_1}t_0$, $Nt_0t_1\bar{t_2}\bar{t_1}\bar{t_0}$ and $Nt_0t_1\bar{t_2}\bar{t_1}t_1$ belong? However,

$Nt_0t_1\bar{t_2}\bar{t_1}t_0 = Nt_0t_1\bar{t_2}\bar{t_1}t_0 \in Nt_0t_1\bar{t_2}\bar{t_1}t_0N$ (New double coset denoted by $[01\bar{2}\bar{1}0]$).
$Nt_0t_1\bar{t_2}\bar{t_1}\bar{t_0} = Nt_0t_1\bar{t_2}\bar{t_1}\bar{t_0} \in Nt_0t_1\bar{t_2}\bar{t_1}\bar{t_0}N$ (New double coset denoted by $[01\bar{2}\bar{1}\bar{0}]$).
$Nt_0t_1\bar{t_2}\bar{t_1}t_1 = Nt_0t_1\bar{t_2} \in [01\bar{2}]$.

Consider the double coset $[01\bar{2}\bar{1}0]$
Claim: $01\bar{2}\bar{1}0 \sim 0\bar{2}120 \sim 02\bar{1}\bar{2}0$

Proof. By using our relation conjugates with elements of N, we have
1. $01\underline{\bar{2}\bar{1}}0 = 01(1,2)(\bar{1},\bar{2})2120 = (1,2)(\bar{1},\bar{2})022120 = (1,2)(\bar{1},\bar{2})0\bar{2}120$.

2. $01\underline{\bar{2}\bar{1}}0 = 01(1,2)(\bar{1},\bar{2})2120 = (1,2)(\bar{1},\bar{2})022120 = (1,2)(\bar{1},\bar{2})0\underline{\bar{2}1}20$
$= (1,2)(\bar{1},\bar{2})0(2,\bar{1})(\bar{2},1)2\bar{1}220 = (1,\bar{1})(\bar{2},2)02\bar{1}\bar{2}0$. $\square$

Since $N^{(01\bar{2}\bar{1}0)} \geq N^{01\bar{2}\bar{1}0} =< e >$ and
$N(t_0t_1\bar{t_2}\bar{t_1}t_0)^{\{(1,\bar{2})(\bar{1},2)\}} = Nt_0\bar{t_2}t_1\bar{t_2}t_0 \Rightarrow (1,\bar{2})(\bar{1},2) \in N^{(01\bar{2}\bar{1}0)}$
$N(t_0t_1\bar{t_2}\bar{t_1}t_0)^{\{(1,2)(\bar{1},\bar{2})\}} = Nt_0t_2\bar{t_1}\bar{t_2}t_0 \Rightarrow (1,2)(\bar{1},\bar{2}) \in N^{(01\bar{2}\bar{1}0)}$
$\Rightarrow N^{(01\bar{2}\bar{1}0)} =< (1,2)(\bar{1},\bar{2}), (1,\bar{2})(\bar{1},2) >$
$= \{e, (1,2)(\bar{1},\bar{2}), (1,\bar{1})(2,\bar{2}), (1,\bar{2})(2,\bar{1})\}$
Therefore, the number of single cosets in double coset $[01\bar{2}\bar{1}0]$ are at most

$\frac{|N|}{|N^{(01\bar{2}\bar{1}0)}|} = \frac{4!}{4} = \frac{24}{4} = 6$. Each single coset has four different names.

Now, take $(01\bar{2}\bar{1}0)$ conjugates with the elements of $N^{(01\bar{2}\bar{1}0)}$, we obtain the following relations:

$$01\bar{2}\bar{1}0 \sim 0\bar{2}120 \sim 02\bar{1}\bar{2}0 \sim 0\bar{1}210$$

By conjugating $(01\bar{2}\bar{1}0 \sim 0\bar{2}120 \sim 02\bar{1}\bar{2}0 \sim 0\bar{1}210)$ with the elements of transversal of $N^{(01\bar{2}\bar{1}0)}$, we obtain the following six single cosets of double coset $[01\bar{2}\bar{1}0]$:

$$01\bar{2}\bar{1}0 \sim 0\bar{2}120 \sim 02\bar{1}\bar{2}0 \sim 0\bar{1}210$$
$$1\bar{0}201 \sim 12\bar{0}\bar{2}1 \sim 1\bar{2}021 \sim 10\bar{2}\bar{0}1$$
$$\bar{0}\bar{1}\bar{2}1\bar{0} \sim \bar{0}\bar{2}\bar{1}2\bar{0} \sim \bar{0}21\bar{2}\bar{0} \sim \bar{0}12\bar{1}\bar{0}$$
$$\bar{2}\bar{0}\bar{1}0\bar{2} \sim \bar{2}\bar{1}\bar{0}1\bar{2} \sim \bar{2}10\bar{1}\bar{2} \sim \bar{2}01\bar{0}\bar{2}$$
$$\bar{1}02\bar{0}\bar{1} \sim \bar{1}20\bar{2}\bar{1} \sim \bar{1}\bar{2}\bar{0}2\bar{1} \sim \bar{1}\bar{0}\bar{2}0\bar{1}$$
$$20\bar{1}\bar{0}2 \sim 2\bar{1}012 \sim 21\bar{0}\bar{1}2 \sim 2\bar{0}102$$

The orbits of $N^{(01\bar{2}\bar{1}0)}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}$, $\{\bar{0}\}$ and $\{1, 2, \bar{1}, \bar{2}\}$. Now, take one representative from each orbit, namely 0, $\bar{0}$ and 1, then find the double coset to which $Nt_0t_1\bar{t_2}\bar{t_1}t_0t_0$, $Nt_0t_1\bar{t_2}\bar{t_1}t_0\bar{t_0}$ and $Nt_0t_1\bar{t_2}\bar{t_1}t_0t_1$ belong? However,

$Nt_0t_1\bar{t_2}\bar{t_1}t_0t_0 = Nt_0t_1\bar{t_2}\bar{t_1}\bar{t_0} \in [01\bar{2}\bar{1}\bar{0}]$.
$Nt_0t_1\bar{t_2}\bar{t_1}t_0\bar{t_0} = Nt_0t_1\bar{t_2}\bar{t_1} \in [01\bar{2}\bar{1}]$.
$Nt_0t_1\bar{t_2}\bar{t_1}t_0t_1 = Nt_0t_1\bar{t_2}\bar{t_1}t_0t_1 \in [0120\bar{1}]$ **(By Lemma-14)**.

Consider the double coset $[01\bar{2}\bar{1}\bar{0}]$
Claim: $01\bar{2}\bar{1}\bar{0} \sim 02\bar{1}\bar{2}\bar{0} \sim 0\bar{1}21\bar{0}$

Proof. By using our relation conjugates with elements of N, we have
1. $01\underline{\bar{2}\bar{1}}\bar{0} = 01(1,2)(\bar{1},\bar{2})212\bar{0} = (1,2)(\bar{1},\bar{2})02212\bar{0} = (1,2)(\bar{1},\bar{2})0\underline{\bar{2}1}2\bar{0}$
$= (1,2)(\bar{1},\bar{2})0(2,\bar{1})(\bar{2},1)2\bar{1}22\bar{0} = (1,\bar{1})(\bar{2},2)02\bar{1}\bar{2}\bar{0}$.

2. $0\underline{1\bar{2}}\bar{1}\bar{0} = 0(2,\bar{1})(\bar{2},1)\bar{1}2\bar{1}\bar{1}\bar{0} = (2,\bar{1})(\bar{2},1)0\bar{1}21\bar{0}$. □

Since $N^{(01\bar{2}\bar{1}\bar{0})} \geq N^{01\bar{2}\bar{1}\bar{0}} =< e >$ and
$N(t_0t_1\bar{t_2}\bar{t_1}\bar{t_0})^{\{(1,2)(\bar{1},\bar{2})\}} = Nt_0t_2\bar{t_1}\bar{t_2}\bar{t_0} \Rightarrow (1,2)(\bar{1},\bar{2}) \in N^{(01\bar{2}\bar{1}\bar{0})}$
$N(t_0t_1\bar{t_2}\bar{t_1}\bar{t_0})^{\{(1,\bar{1})(2,\bar{2})\}} = Nt_0\bar{t_1}t_2t_1\bar{t_0} \Rightarrow (1,\bar{1})(2,\bar{2}) \in N^{(01\bar{2}\bar{1}\bar{0})}$

$\Rightarrow N^{(01\bar{2}\bar{1}\bar{0})} =< (1,2)(\bar{1},\bar{2}),(1,\bar{1})(2,\bar{2}) >$
$= \{e,(1,2)(\bar{1},\bar{2}),(1,\bar{1})(2,\bar{2}),(1,\bar{2})(2,\bar{1})\}$
Therefore, the number of single cosets in the double coset $[01\bar{2}\bar{1}\bar{0}]$ are at most

$\frac{|N|}{|N^{(01\bar{2}\bar{1}\bar{0})}|}=\frac{4!}{4}=\frac{24}{4}=6$. Each single coset has four different names.

Now, take $(01\bar{2}\bar{1}\bar{0})$ conjugates with the elements of $N^{(01\bar{2}\bar{1}\bar{0})}$, we obtain the following relations:

$$01\bar{2}\bar{1}\bar{0} \sim 02\bar{1}\bar{2}\bar{0} \sim 0\bar{1}21\bar{0} \sim 0\bar{2}12\bar{0}$$

By conjugating $(01\bar{2}\bar{1}\bar{0} \sim 02\bar{1}\bar{2}\bar{0} \sim 0\bar{1}21\bar{0} \sim 0\bar{2}12\bar{0})$ with the elements of transversal of $N^{(01\bar{2}\bar{1}\bar{0})}$, we obtain the following six single cosets of double coset $[01\bar{2}\bar{1}\bar{0}]$:

$$\begin{array}{c}
01\bar{2}\bar{1}\bar{0} \sim 02\bar{1}\bar{2}\bar{0} \sim 0\bar{1}21\bar{0} \sim 0\bar{2}12\bar{0}\\
1\bar{0}20\bar{1} \sim 1\bar{2}02\bar{1} \sim 10\bar{2}\bar{0}\bar{1} \sim 12\bar{0}\bar{2}\bar{1}\\
\bar{0}\bar{1}\bar{2}10 \sim \bar{0}21\bar{2}0 \sim \bar{0}12\bar{1}0 \sim \bar{0}\bar{2}\bar{1}20\\
\bar{2}\bar{0}\bar{1}02 \sim \bar{2}10\bar{1}2 \sim \bar{2}01\bar{0}2 \sim \bar{2}\bar{1}\bar{0}12\\
\bar{1}02\bar{0}1 \sim \bar{1}\bar{2}\bar{0}21 \sim \bar{1}\bar{0}\bar{2}01 \sim \bar{1}20\bar{2}1\\
20\bar{1}\bar{0}\bar{2} \sim 21\bar{0}\bar{1}\bar{2} \sim 2\bar{0}10\bar{2} \sim 2\bar{1}01\bar{2}
\end{array}$$

The orbits of $N^{(01\bar{2}\bar{1}\bar{0})}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{\bar{0}\}$ and $\{1,2,\bar{1},\bar{2}\}$. Now, take one representative from each orbit, namely 0, $\bar{0}$ and 1, then find the double coset to which $Nt_0t_1\bar{t_2}\bar{t_1}\bar{t_0}t_0$, $Nt_0t_1\bar{t_2}\bar{t_1}\bar{t_0}\bar{t_0}$ and $Nt_0t_1\bar{t_2}\bar{t_1}\bar{t_0}t_1$ belong? However,

$Nt_0t_1\bar{t_2}\bar{t_1}\bar{t_0}t_0 = Nt_0t_1\bar{t_2}\bar{t_1} \in [01\bar{2}\bar{1}]$.
$Nt_0t_1\bar{t_2}\bar{t_1}\bar{t_0}\bar{t_0} = Nt_0t_1\bar{t_2}\bar{t_1}t_0 \in [01\bar{2}\bar{1}0]$.
$Nt_0t_1\bar{t_2}\bar{t_1}\bar{t_0}t_1 = Nt_0t_1\bar{t_2}\bar{t_1}\bar{t_0}t_1 \in [01\bar{2}01]$ **(By Lemma-15).**

Consider the double coset $[01\bar{2}01]$.

$N^{(01\bar{2}01)} = N^{01\bar{2}01} =< e >$.
Therefore, the number of single cosets in the double coset $[01\bar{2}01]$ are at most

$\frac{|N|}{|N^{(01\bar{2}01)}|}=\frac{4!}{1}=\frac{24}{1}=24$. Each single coset has only one name.

By conjugating $(01\bar{2}01)$ with the elements of transversal of $N^{(01\bar{2}01)}$, we obtain the following 24 single cosets of double coset $[01\bar{2}01]$:
$01\bar{2}01, \bar{2}10\bar{2}1, 21\bar{0}21, \bar{0}12\bar{0}1, 12\bar{0}12, \bar{1}20\bar{1}2, \bar{0}21\bar{0}2, 02\bar{1}02, \bar{1}02\bar{1}0, 10\bar{2}10, 20\bar{1}20, \bar{2}01\bar{2}0,$
$\bar{0}\bar{1}2\bar{0}\bar{1}, 0\bar{1}20\bar{1}, \bar{2}\bar{1}\bar{0}\bar{2}\bar{1}, 2\bar{1}02\bar{1}, 0\bar{2}10\bar{2}, \bar{0}\bar{2}\bar{1}\bar{0}\bar{2}, 1\bar{2}01\bar{2}, \bar{1}\bar{2}\bar{0}\bar{1}\bar{2}, 1\bar{0}21\bar{0}, \bar{2}\bar{0}\bar{1}\bar{2}\bar{0}, \bar{1}\bar{0}\bar{2}\bar{1}\bar{0}, 2\bar{0}12\bar{0}.$

The orbits of $N^{(01\bar{2}01)}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1}\}$ and $\{\bar{2}\}$. Now, take one representative from each orbit, namely 0, 1, 2, $\bar{0}$, $\bar{1}$ and $\bar{2}$, then find the double cosets to which $Nt_0t_1\bar{t_2}t_0t_1t_0$, $Nt_0t_1\bar{t_2}t_0t_1t_1$, $Nt_0t_1\bar{t_2}t_0t_1t_2$, $Nt_0t_1\bar{t_2}t_0t_1\bar{t_0}$, $Nt_0t_1\bar{t_2}t_0t_1\bar{t_1}$ and $Nt_0t_1\bar{t_2}t_0t_1\bar{t_2}$ belong? However,

$Nt_0t_1\bar{t_2}t_0t_1t_0 = Nt_0t_1\bar{t_2}t_0t_1t_0 \in [01\bar{2}\bar{1}\bar{0}]$ **(By Lemma-16)**.
$Nt_0t_1\bar{t_2}t_0t_1t_1 = Nt_0t_1\bar{t_2}t_0\bar{t_1} \in [01\bar{2}\bar{0}1]$ **(By Lemma-17)**.
$Nt_0t_1\bar{t_2}t_0t_1t_2 = Nt_0t_1\bar{t_2}t_0t_1t_2 \in [01\bar{2}01]$ **(By Lemma-18)**.
$Nt_0t_1\bar{t_2}t_0t_1\bar{t_0} = Nt_0t_1\bar{t_2}t_0t_1\bar{t_0} \in [01\bar{2}01]$ **(By Lemma-19)**.
$Nt_0t_1\bar{t_2}t_0t_1\bar{t_1} = Nt_0t_1\bar{t_2}t_0 \in [01\bar{2}0]$.
$Nt_0t_1\bar{t_2}t_0t_1\bar{t_2} = Nt_0t_1\bar{t_2}t_0t_1\bar{t_2} \in Nt_0t_1\bar{t_2}t_0t_1\bar{t_2}N$ (New double coset denoted by $[01\bar{2}01\bar{2}]$).

Consider the new double coset $[012\bar{1}\bar{0}]$.
Claim: $012\bar{1}\bar{0} \sim 0\bar{1}\bar{2}1\bar{0} \sim 021\bar{2}\bar{0}$

Proof. By using our relation conjugates with elements of N, we have
1. $0\underline{12}\bar{1}\bar{0} = 0(1,2)(\bar{1},\bar{2})\bar{1}\bar{2}\bar{1}1\bar{0} = (1,2)(\bar{1},\bar{2})0\bar{1}\bar{2}1\bar{0}$.

2. $01\underline{2\bar{1}}\bar{0} = 01(1,\bar{2})(\bar{1},2)\bar{2}1\bar{2}\bar{0} = (1,\bar{2})(\bar{1},2)0\bar{2}\bar{2}1\bar{2}\bar{0} = (1,\bar{2})(\bar{1},2)021\bar{2}\bar{0}$. ☐

Since $N^{(012\bar{1}\bar{0})} \geq N^{012\bar{1}\bar{0}} =< e >$ and
$N(t_0t_1t_2\bar{t_1}\bar{t_0})^{\{(1,\bar{1})(\bar{2},2)\}} = Nt_0\bar{t_1}\bar{t_2}t_1\bar{t_0} \Rightarrow (1,\bar{1})(\bar{2},2) \in N^{(012\bar{1}\bar{0})}$
$N(t_0t_1t_2\bar{t_1}\bar{t_0})^{\{(1,2)(\bar{1},\bar{2})\}} = Nt_0t_2t_1\bar{t_2}\bar{t_0} \Rightarrow (1,2)(\bar{1},\bar{2}) \in N^{(012\bar{1}\bar{0})}$
$\Rightarrow N^{(012\bar{1}\bar{0})} =< (1,\bar{1})(\bar{2},2), (1,2)(\bar{1},\bar{2}) >$
$= \{e, (1,\bar{2})(2,\bar{1}), (1,2)(\bar{1},\bar{2}), (1,\bar{1})(2,\bar{2})\}$.
Therefore, the number of single cosets in the double coset $[012\bar{1}\bar{0}]$ are at most

$\frac{|N|}{|N^{(012\bar{1}\bar{0})}|} = \frac{4!}{4} = \frac{24}{4} = 6$. Each single coset has four different names.

Now, take $(012\bar{1}\bar{0})$ conjugates with the elements of $N^{(012\bar{1}\bar{0})}$, we obtain the following relations:

$$012\bar{1}\bar{0} \sim 0\bar{2}\bar{1}2\bar{0} \sim 021\bar{2}\bar{0} \sim 0\bar{1}\bar{2}1\bar{0}$$

By conjugating $(012\bar{1}\bar{0} \sim 0\bar{2}\bar{1}2\bar{0} \sim 021\bar{2}\bar{0} \sim 0\bar{1}\bar{2}1\bar{0})$ with the elements of transversal of $N^{(012\bar{1}\bar{0})}$, we obtained the six single cosets of double coset $[012\bar{1}\bar{0}]$:

$$\begin{array}{c}
012\bar{1}\bar{0} \sim 0\bar{2}\bar{1}2\bar{0} \sim 021\bar{2}\bar{0} \sim 0\bar{1}\bar{2}1\bar{0} \\
1\bar{0}\bar{2}0\bar{1} \sim 120\bar{2}\bar{1} \sim 1\bar{2}\bar{0}2\bar{1} \sim 102\bar{0}\bar{1} \\
\bar{0}\bar{1}210 \sim \bar{0}\bar{2}120 \sim \bar{0}2\bar{1}\bar{2}0 \sim \bar{0}1\bar{2}\bar{1}0 \\
\bar{2}\bar{0}102 \sim \bar{2}\bar{1}012 \sim \bar{2}1\bar{0}\bar{1}2 \sim \bar{2}0\bar{1}\bar{0}2 \\
\bar{1}0\bar{2}\bar{0}1 \sim \bar{1}2\bar{0}\bar{2}1 \sim \bar{1}\bar{2}021 \sim \bar{1}\bar{0}201 \\
2\bar{0}\bar{1}0\bar{2} \sim 210\bar{1}\bar{2} \sim 2\bar{1}\bar{0}1\bar{2} \sim 201\bar{0}\bar{2}
\end{array}$$

The orbits of $N^{(012\bar{1}\bar{0})}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{\bar{0}\}$ and $\{1,2,\bar{1},\bar{2}\}$. Now, take one representative from each orbit, namely 0, $\bar{0}$ and 1, then find the double coset to which $Nt_0t_1t_2\bar{t_1}\bar{t_0}t_0$, $Nt_0t_1t_2\bar{t_1}\bar{t_0}\bar{t_0}$ and $Nt_0t_1t_2\bar{t_1}\bar{t_0}t_1$ belong? However,

$Nt_0t_1t_2\bar{t_1}\bar{t_0}t_0 = Nt_0t_1t_2\bar{t_1} \in [012\bar{1}]$.
$Nt_0t_1t_2\bar{t_1}\bar{t_0}\bar{t_0} = Nt_0t_1t_2\bar{t_1}t_0 \in [012\bar{1}0]$.
$Nt_0t_1t_2\bar{t_1}\bar{t_0}t_1 = Nt_0t_1t_2\bar{t_1}\bar{t_0}t_1 \in [01201]$ **(By Lemma-20).**

Consider the new double coset $[012\bar{1}0]$.
Claim: $012\bar{1}0 \sim 0\bar{2}\bar{1}20 \sim 021\bar{2}0$

Proof. By using our relation conjugates with elements of N, we have
1. $01\underline{2\bar{1}}0 = 01(\bar{2},1)(2,\bar{1})\bar{2}1\bar{2}0 = (\bar{2},1)(2,\bar{1})0\bar{2}\bar{2}1\bar{2}0 = (\bar{2},1)(2,\bar{1})021\bar{2}0$
$= (\bar{2},1)(2,\bar{1})0\underline{21}220 = (\bar{2},1)(2,\bar{1})0(1,2)(\bar{1},\bar{2})\bar{2}\bar{1}20 = (1,\bar{1})(2,\bar{2})0\bar{2}\bar{1}20.$

2. $01\underline{2\bar{1}}0 = 01(\bar{2},1)(2,\bar{1})\bar{2}1\bar{2}0 = (\bar{2},1)(2,\bar{1})0\bar{2}\bar{2}1\bar{2}0 = (\bar{2},1)(2,\bar{1})021\bar{2}0.$ ∎

Since $N^{(012\bar{1}0)} \geq N^{012\bar{1}0} =< e >$ and
$N(t_0t_1t_2\bar{t_1}t_0)^{\{(1,\bar{2})(\bar{1},2)\}} = Nt_0\bar{t_2}\bar{t_1}t_2t_0 \Rightarrow (1,\bar{2})(\bar{1},2) \in N^{(012\bar{1}0)}$
$N(t_0t_1t_2\bar{t_1}t_0)^{\{(1,2)(\bar{1},\bar{2})\}} = Nt_0t_2t_1\bar{t_2}t_0 \Rightarrow (1,2)(\bar{1},\bar{2}) \in N^{(012\bar{1}0)}$
$\Rightarrow N^{(012\bar{1}0)} =< (1,\bar{2})(\bar{1},2), (1,2)(\bar{1},\bar{2}) >$

$= \{e, (1,\bar{2})(2,\bar{1}), (1,2)(\bar{1},\bar{2}), (1,\bar{1})(2,\bar{2})\}$.

Therefore, the number of single cosets in the double coset $[012\bar{1}0]$ are at most

$\frac{|N|}{|N^{(012\bar{1}0)}|}=\frac{4!}{4} = \frac{24}{4} = 6$. Each single coset has four different names.

Now, take $(012\bar{1}0)$ conjugates with the elements of $N^{(012\bar{1}0)}$, we obtain the following relations:

$$012\bar{1}0 \sim 0\bar{2}\bar{1}20 \sim 021\bar{2}0 \sim 0\bar{1}\bar{2}10$$

By conjugating $(012\bar{1}0 \sim 0\bar{2}\bar{1}20 \sim 021\bar{2}0 \sim 0\bar{1}\bar{2}10)$ with the elements of transversal of $N^{(012\bar{1}0)}$, we obtained the following six single cosets of double coset $[012\bar{1}0]$:

$$\begin{array}{l}
012\bar{1}0 \sim 0\bar{2}\bar{1}20 \sim 021\bar{2}0 \sim 0\bar{1}\bar{2}10 \\
1\bar{0}\bar{2}01 \sim 120\bar{2}1 \sim 1\bar{2}\bar{0}21 \sim 1020\bar{1} \\
\bar{0}\bar{1}21\bar{0} \sim \bar{0}\bar{2}12\bar{0} \sim \bar{0}2\bar{1}\bar{2}\bar{0} \sim \bar{0}1\bar{2}\bar{1}\bar{0} \\
\bar{2}\bar{0}10\bar{2} \sim \bar{2}\bar{1}01\bar{2} \sim \bar{2}1\bar{0}\bar{1}\bar{2} \sim \bar{2}0\bar{1}\bar{0}\bar{2} \\
\bar{1}0\bar{2}\bar{0}\bar{1} \sim \bar{1}2\bar{0}\bar{2}\bar{1} \sim \bar{1}\bar{2}02\bar{1} \sim \bar{1}\bar{0}20\bar{1} \\
201\bar{0}2 \sim 2\bar{1}\bar{0}12 \sim 210\bar{1}2 \sim 2\bar{0}\bar{1}02
\end{array}$$

The orbits of $N^{(012\bar{1}0)}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{\bar{0}\}$ and $\{1,2,\bar{1},\bar{2}\}$. Now, take one representative from each orbit, namely 0, $\bar{0}$ and 1, then find the double coset to which $Nt_0t_1t_2\bar{t_1}t_0t_0$, $Nt_0t_1t_2\bar{t_1}t_0\bar{t_0}$ and $Nt_0t_1t_2\bar{t_1}t_0t_1$ belong? However,

$Nt_0t_1t_2\bar{t_1}t_0t_0 = Nt_0t_1t_2\bar{t_1}\bar{t_0} \in [012\bar{1}\bar{0}]$.
$Nt_0t_1t_2\bar{t_1}t_0\bar{t_0} = Nt_0t_1t_2\bar{t_1} \in [012\bar{1}]$.
$Nt_0t_1t_2\bar{t_1}t_0t_1 = Nt_0t_1t_2\bar{t_1}t_0t_1 \in [[0120\bar{1}]$ **(By Lemma-21)**.

Consider the new double coset $[012\bar{0}1]$.

$N^{(012\bar{0}1)} = N^{012\bar{0}1} =< e >$.

Therefore, the number of single cosets in the double coset $[012\bar{0}1]$ are at most

$\frac{|N|}{|N^{(012\bar{0}1)}|}=\frac{4!}{1} = \frac{24}{1} = 24$. Each single coset has only one name.

By conjugating $(012\bar{0}1)$ with the elements of transversal of $N^{(012\bar{0}1)}$, we obtain the 24 single cosets of double coset $[012\bar{0}1]$:
$012\bar{0}1, \bar{2}1\bar{0}21, 210\bar{2}1, \bar{0}1\bar{2}01, 120\bar{1}2, \bar{1}2\bar{0}12, \bar{0}2\bar{1}02, 021\bar{0}2, \bar{1}0\bar{2}10, 102\bar{1}0, 201\bar{2}0, \bar{2}0\bar{1}20,$
$\bar{0}120\bar{1}, 0\bar{1}\bar{2}0\bar{1}, \bar{2}\bar{1}02\bar{1}, 2\bar{1}\bar{0}2\bar{1}, 0\bar{2}\bar{1}0\bar{2}, \bar{0}210\bar{2}, 1\bar{2}0\bar{1}\bar{2}, \bar{1}201\bar{2}, 1\bar{0}2\bar{1}\bar{0}, \bar{2}012\bar{0}, \bar{1}021\bar{0}, 2\bar{0}\bar{1}2\bar{0}.$

The orbits of $N^{(012\bar{0}1)}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1}\}$ and $\{\bar{2}\}$. Now, take one representative from each orbit, namely 0, 1, 2, $\bar{0}$, $\bar{1}$ and $\bar{2}$, then find the double cosets to which $Nt_0t_1t_2\bar{t_0}t_1t_0$, $Nt_0t_1t_2\bar{t_0}t_1t_1$, $Nt_0t_1t_2\bar{t_0}t_1t_2$, $Nt_0t_1t_2\bar{t_0}t_1\bar{t_0}$, $Nt_0t_1t_2\bar{t_0}t_1\bar{t_1}$ and $Nt_0t_1t_2\bar{t_0}t_1\bar{t_2}$ belong? However,

$Nt_0t_1t_2\bar{t_0}t_1t_0 = Nt_0t_1t_2\bar{t_0}t_1t_0 \in [012\bar{0}1]$ **(By Lemma-22).**
$Nt_0t_1t_2\bar{t_0}t_1t_1 = Nt_0t_1t_2\bar{t_0}\bar{t_1} \in [0120\bar{1}]$ **(By Lemma-7).**
$Nt_0t_1t_2\bar{t_0}t_1t_2 = Nt_0t_1t_2\bar{t_0}t_1t_2 \in [01\bar{2}01]$ **(By Lemma-23).**
$Nt_0t_1t_2\bar{t_0}t_1\bar{t_0} = Nt_0t_1t_2\bar{t_0}t_1\bar{t_0} \in [012\bar{1}0]$ **(By Lemma-24).**
$Nt_0t_1t_2\bar{t_0}t_1\bar{t_1} = Nt_0t_1t_2\bar{t_0} \in [012\bar{0}]$.
$Nt_0t_1t_2\bar{t_0}t_1\bar{t_2} = Nt_0t_1t_2\bar{t_0}t_1\bar{t_2} \in [01\bar{2}0]$ **(By Lemma-25).**

Consider the new double coset $[0120\bar{1}]$.

$N^{(0120\bar{1})} = N^{0120\bar{1}} = < e >$.
Therefore, the number of single cosets in the double coset $[0120\bar{1}]$ are at most

$\frac{|N|}{|N^{(01201)}|} = \frac{4!}{1} = \frac{24}{1} = 24$. Each single coset has only one name.

By conjugating $(0120\bar{1})$ with the elements of transversal of $N^{(0120\bar{1})}$, we obtain the following 24 single cosets of double coset $[0120\bar{1}]$:
$0120\bar{1}, \bar{2}1\bar{0}2\bar{1}, 2102\bar{1}, \bar{0}1\bar{2}0\bar{1}, 1201\bar{2}, \bar{1}2\bar{0}1\bar{2}, \bar{0}2\bar{1}0\bar{2}, 0210\bar{2}, \bar{1}0\bar{2}1\bar{0}, 1021\bar{0}, 2012\bar{0}, \bar{2}0\bar{1}2\bar{0},$
$\bar{0}\bar{1}2\bar{0}1, 0\bar{1}\bar{2}01, \bar{2}\bar{1}0\bar{2}1, 2\bar{1}\bar{0}21, 0\bar{2}\bar{1}02, \bar{0}\bar{2}1\bar{0}2, 1\bar{2}\bar{0}12, \bar{1}\bar{2}0\bar{1}2, 1\bar{0}\bar{2}10, \bar{2}\bar{0}1\bar{2}0, \bar{1}\bar{0}2\bar{1}0, 2\bar{0}\bar{1}20.$

The orbits of $N^{(0120\bar{1})}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1}\}$ and $\{\bar{2}\}$. Now, take one representative from each orbit, namely 0, 1, 2, $\bar{0}$, $\bar{1}$ and $\bar{2}$, then find the double cosets to which $Nt_0t_1t_2t_0\bar{t_1}t_0$, $Nt_0t_1t_2t_0\bar{t_1}t_1$, $Nt_0t_1t_2t_0\bar{t_1}t_2$, $Nt_0t_1t_2t_0\bar{t_1}\bar{t_0}$, $Nt_0t_1t_2t_0\bar{t_1}\bar{t_1}$ and $Nt_0t_1t_2t_0\bar{t_1}\bar{t_2}$ belong? However,

$Nt_0t_1t_2t_0\bar{t_1}t_0 = Nt_0t_1t_2t_0\bar{t_1}t_0 \in [012\bar{0}]$ **(By Lemma-26)**.
$Nt_0t_1t_2t_0\bar{t_1}t_1 = Nt_0t_1t_2t_0 \in [0120]$.
$Nt_0t_1t_2t_0\bar{t_1}t_2 = Nt_0t_1t_2t_0\bar{t_1}t_2 \in [01\bar{2}\bar{1}0]$ **(By Lemma-27)**.
$Nt_0t_1t_2t_0\bar{t_1}\bar{t_0} = Nt_0t_1t_2t_0\bar{t_1}\bar{t_0} \in [012\bar{0}1]$ **(By Lemma-28)**.
$Nt_0t_1t_2t_0\bar{t_1}\bar{t_1} = Nt_0t_1t_2t_0t_1 \in [01201]$.
$Nt_0t_1t_2t_0\bar{t_1}\bar{t_2} = Nt_0t_1t_2t_0\bar{t_1}\bar{t_2} \in [0120\bar{1}]$ **(By Lemma-29)**.

Consider the new double coset [01201].

$N^{(01201)} = N^{01201} =< e >$.
Therefore, the number of single cosets in the double coset [01201] are at most

$$\frac{|N|}{|N^{(01201)}|}=\frac{4!}{1} = \frac{24}{1} = 24.$$ Each single coset has only one name.

By conjugating (01201) with the elements of transversal of $N^{(01201)}$, we obtain the 24 single cosets of double coset [01201]:
$01201, \bar{2}1\bar{0}\bar{2}1, 21021, \bar{0}1\bar{2}\bar{0}1, 12012, \bar{1}2\bar{0}\bar{1}2, \bar{0}2\bar{1}\bar{0}2, 02102, \bar{1}0\bar{2}\bar{1}0, 10210, 20120, \bar{2}0\bar{1}\bar{2}0,$
$\bar{0}\bar{1}2\bar{0}\bar{1}, 0\bar{1}\bar{2}0\bar{1}, \bar{2}\bar{1}0\bar{2}\bar{1}, 2\bar{1}\bar{0}2\bar{1}, 0\bar{2}\bar{1}0\bar{2}, \bar{0}2\bar{1}\bar{0}\bar{2}, 1\bar{2}\bar{0}1\bar{2}, \bar{1}\bar{2}0\bar{1}\bar{2}, 1\bar{0}\bar{2}1\bar{0}, \bar{2}\bar{0}1\bar{2}\bar{0}, \bar{1}\bar{0}2\bar{1}\bar{0}, 2\bar{0}\bar{1}2\bar{0}.$

The orbits of $N^{(01201)}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1}\}$ and $\{\bar{2}\}$. Now, take one representative from each orbit, namely 0, 1, 2, $\bar{0}$, $\bar{1}$ and $\bar{2}$, then find the double cosets to which $Nt_0t_1t_2t_0t_1t_0$, $Nt_0t_1t_2t_0t_1t_1$, $Nt_0t_1t_2t_0t_1t_2$, $Nt_0t_1t_2t_0t_1\bar{t_0}$, $Nt_0t_1t_2t_0t_1\bar{t_1}$ and $Nt_0t_1t_2t_0t_1\bar{t_2}$ belong? However,

$Nt_0t_1t_2t_0t_1t_0 = Nt_0t_1t_2t_0t_1t_0 \in [012\bar{1}\bar{0}]$ **(By Lemma-30)**.
$Nt_0t_1t_2t_0t_1t_1 = Nt_0t_1t_2t_0\bar{t_1} \in [0120\bar{1}]$.
$Nt_0t_1t_2t_0t_1t_2 = Nt_0t_1t_2t_0t_1t_2 \in Nt_0t_1t_2t_0t_1t_2N$ (New double coset denote by [012012]).
$Nt_0t_1t_2t_0t_1\bar{t_0} = Nt_0t_1t_2t_0t_1\bar{t_0} \in [01201]$ **(By Lemma-31)**.
$Nt_0t_1t_2t_0t_1\bar{t_1} = Nt_0t_1t_2t_0 \in [0120]$.
$Nt_0t_1t_2t_0t_1\bar{t_2} = Nt_0t_1t_2t_0t_1\bar{t_2} \in [01201]$ **(By Lemma-32)**.

Consider the new double coset [012012].

Claim: $012012 \sim 0\bar{2}\bar{1}0\bar{2}\bar{1} \sim 021021$

Proof. By using our relation conjugates with elements of N, we have

1. $0120\underline{12} = 0120(1,2)(\bar{1},\bar{2})\bar{1}\bar{2}\bar{1} = (1,2)(\bar{1},\bar{2})0210\bar{1}\bar{2}\bar{1} = (1,2)(\bar{1},\bar{2})02\underline{101}1\bar{2}\bar{1}$
$= (1,2)(\bar{1},\bar{2})02(1,0)(\bar{1},\bar{0})\bar{1}\bar{0}1\bar{2}\bar{1} = (1,2,0)(\bar{1},\bar{2},\bar{0})12\bar{1}\underline{\bar{0}1}\bar{2}\bar{1}$
$= (1,2,0)(\bar{1},\bar{2},\bar{0})12\bar{1}(0,\bar{1})(\bar{0},1)0\bar{1}0\bar{2}\bar{1} = (1,2,\bar{1},\bar{2})(0,\bar{0})\bar{0}200\bar{1}0\bar{2}\bar{1}$
$= (1,2,\bar{1},\bar{2})(0,\bar{0})\underline{\bar{0}2\bar{0}}\bar{1}0\bar{2}\bar{1} = (1,2,\bar{1},\bar{2})(0,\bar{0})(\bar{0},2)(0,\bar{2})0\bar{2}\bar{1}0\bar{2}\bar{1}$
$= (1,\bar{0},\bar{2})(\bar{1},0,2)0\bar{2}\bar{1}0\bar{2}\bar{1}.$

2. $0120\underline{12} = 0120(1,2)(\bar{1},\bar{2})\bar{1}\bar{2}\bar{1} = (1,2)(\bar{1},\bar{2})0210\bar{1}\bar{2}\bar{1} = (1,2)(\bar{1},\bar{2})0210\bar{1}\underline{\bar{2}1}1$
$= (1,2)(\bar{1},\bar{2})0210\bar{1}(\bar{1},2)(1,\bar{2})2\bar{1}21 = (1,\bar{1})(2,\bar{2})0\bar{1}\bar{2}022\bar{1}21$
$= (1,\bar{1})(2,\bar{2})0\bar{1}\underline{\bar{2}0\bar{2}}\bar{1}21 = (1,\bar{1})(2,\bar{2})0\bar{1}(0,\bar{2})(2,\bar{0})2\bar{0}\bar{1}21$
$= (1,\bar{1})(2,0,\bar{2},\bar{0})\bar{2}\underline{\bar{1}2}\bar{0}\bar{1}21 = (1,\bar{1})(2,0,\bar{2},\bar{0})\bar{2}(1,\bar{2})(2,\bar{1})1\bar{2}1\bar{0}\bar{1}21$
$= (1,2,0)(\bar{1},\bar{2},\bar{0})11\bar{2}1\bar{0}\bar{1}21 = (1,2,0)(\bar{1},\bar{2},\bar{0})\bar{1}\bar{2}1\underline{\bar{0}\bar{1}}21$
$= (1,2,0)(\bar{1},\bar{2},\bar{0})\bar{1}\bar{2}1(0,1)(\bar{0},\bar{1})01021 = (1,2)(\bar{1},\bar{2})\bar{0}\bar{2}001021$
$= (1,2)(\bar{1},\bar{2})\underline{\bar{0}\bar{2}\bar{0}}1021 = (1,2)(\bar{1},\bar{2})(0,2)(\bar{0},\bar{2})021021$
$= (1,0,2)(\bar{1},\bar{0},\bar{2})021021.$ □

Since $N^{(012012)} \geq N^{012012} =< e >$ and
$N(t_0t_1t_2t_0t_1t_2)^{\{(1,\bar{2})(\bar{1},2)\}} = Nt_0\bar{t_2}\bar{t_1}t_0\bar{t_2}\bar{t_1} \Rightarrow (1,\bar{2})(\bar{1},2) \in N^{(012012)}$
$N(t_0t_1t_2t_0t_1t_2)^{\{(1,2)(\bar{1},\bar{2})\}} = Nt_0t_2t_1t_0t_2t_1 \Rightarrow (1,2)(\bar{1},\bar{2}) \in N^{(012012)}$
$\Rightarrow N^{(012012)} =< (1,\bar{2})(\bar{1},2), (1,2)(\bar{1},\bar{2}) >$
$= \{e, (1,\bar{2})(2,\bar{1}), (1,2)(\bar{1},\bar{2}), (1,\bar{1})(2,\bar{2})\}.$
Therefore, the number of single cosets in the double coset [012012] are at most

$\frac{|N|}{|N^{(012012)}|} = \frac{4!}{4} = \frac{24}{4} = 6$. Each single coset has four different names.

Now, take (012012) conjugates with the elements of $N^{(012012)}$, we obtain the following relations:

$$012012 \sim 0\bar{2}\bar{1}0\bar{2}\bar{1} \sim 021021 \sim 0\bar{1}\bar{2}0\bar{1}\bar{2}$$

By conjugating ($012012 \sim 0\bar{2}\bar{1}0\bar{2}\bar{1} \sim 021021 \sim 0\bar{1}\bar{2}0\bar{1}\bar{2}$) with the elements of transversal of $N^{(012012)}$, we obtain the following six single cosets of double coset [012012]:

$$012012 \sim 0\bar{2}\bar{1}0\bar{2}\bar{1} \sim 021021 \sim 0\bar{1}\bar{2}0\bar{1}\bar{2}$$
$$1\bar{0}\bar{2}1\bar{0}\bar{2} \sim 120120 \sim 1\bar{2}\bar{0}1\bar{2}\bar{0} \sim 102102$$

$$\bar{0}\bar{1}2\bar{0}\bar{1}2 \sim \bar{0}\bar{2}1\bar{0}\bar{2}1 \sim \bar{0}2\bar{1}\bar{0}2\bar{1} \sim \bar{0}1\bar{2}\bar{0}1\bar{2}$$
$$\bar{2}\bar{0}1\bar{2}\bar{0}1 \sim \bar{2}\bar{1}0\bar{2}\bar{1}0 \sim \bar{2}1\bar{0}\bar{2}1\bar{0} \sim \bar{2}0\bar{1}\bar{2}0\bar{1}$$
$$\bar{1}0\bar{2}\bar{1}0\bar{2} \sim \bar{1}2\bar{0}\bar{1}2\bar{0} \sim \bar{1}\bar{2}0\bar{1}\bar{2}0 \sim \bar{1}\bar{0}2\bar{1}\bar{0}2$$
$$2\bar{0}\bar{1}2\bar{0}\bar{1} \sim 210210 \sim 2\bar{1}\bar{0}2\bar{1}\bar{0} \sim 201201$$

The orbits of $N^{(012012)}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{\bar{0}\}$ and $\{1,2,\bar{1},\bar{2}\}$. Now, take one representative from each orbit, namely 0, $\bar{0}$ and $\bar{2}$, then find the double coset to which $Nt_0t_1t_2t_0t_1t_2t_0$, $Nt_0t_1t_2t_0t_1t_2\bar{t_0}$ and $Nt_0t_1t_2t_0t_1t_2\bar{t_2}$ belong? However,

$Nt_0t_1t_2t_0t_1t_2t_0 = Nt_0t_1t_2t_0t_1t_2t_0 \in Nt_0t_1t_2t_0t_1t_2t_0N$ (New, denoted by [0120120]).
$Nt_0t_1t_2t_0t_1t_2\bar{t_0} = Nt_0t_1t_2t_0t_1t_2\bar{t_0} \in [012012]$ **(By Lemma-33)**.
$Nt_0t_1t_2t_0t_1t_2\bar{t_2} = Nt_0t_1t_2t_0t_1 \in [01201]$.

Consider the new double coset $[01\bar{2}01\bar{2}]$.

Claim: $01\bar{2}01\bar{2} \sim 0\bar{1}20\bar{1}2 \sim 02\bar{1}02\bar{1}$

Proof. By using our relation conjugates with elements of N, we have
1. $01\underline{\bar{2}0}1\bar{2} = 01(\bar{2},0)(2,\bar{0})2\bar{0}21\bar{2} = (\bar{2},0)(2,\bar{0})\bar{2}12\bar{0}21\bar{2} = (\bar{2},0)(2,\bar{0})\bar{2}12\bar{0}\underline{212}2$
$= (\bar{2},0)(2,\bar{0})\bar{2}12\bar{0}(1,2)(\bar{1},\bar{2})\bar{2}\bar{1}2 = (\bar{2},0,\bar{1})(2,\bar{0},1)\bar{1}21\underline{\bar{0}\bar{2}}\bar{1}2$
$= (\bar{2},0,\bar{1})(2,\bar{0},1)\bar{1}21(0,2)(\bar{0},\bar{2})020\bar{1}2 = (\bar{0},1,0,\bar{1})(2,\bar{2})\bar{1}\underline{010}20\bar{1}2$
$= (\bar{0},1,0,\bar{1})(2,\bar{2})\bar{1}(0,1)(\bar{0},\bar{1})\bar{0}\bar{1}20\bar{1}2 = (0,\bar{0})(2,\bar{2})\bar{0}\bar{0}\bar{1}20\bar{1}2 = (0,\bar{0})(2,\bar{2})0\bar{1}20\bar{1}2.$

2. $01\bar{2}0\underline{1\bar{2}} = 01\bar{2}0(1,\bar{2})(\bar{1},2)\bar{1}2\bar{1} = (1,\bar{2})(\bar{1},2)0\bar{2}10\bar{1}2\bar{1} = (1,\bar{2})(\bar{1},2)0\bar{2}\underline{101}12\bar{1}$
$= (1,\bar{2})(\bar{1},2)0\bar{2}(1,0)(\bar{1},\bar{0})\bar{1}\bar{0}12\bar{1} = (1,\bar{2},0)(\bar{1},2,\bar{0})1\bar{2}\bar{1}\underline{\bar{0}1}2\bar{1}$
$= (1,\bar{2},0)(\bar{1},2,\bar{0})1\bar{2}\bar{1}(\bar{0},1)(0,\bar{1})0\bar{1}02\bar{1} = (1,\bar{2},\bar{1},2)(0,\bar{0})\bar{0}\bar{2}00\bar{1}02\bar{1}$
$= (1,\bar{2},\bar{1},2)(0,\bar{0})\underline{\bar{0}\bar{2}\bar{0}}\bar{1}02\bar{1} = (1,\bar{2},\bar{1},2)(0,\bar{0})(0,2)(\bar{0},\bar{2})02\bar{1}02\bar{1}$
$= (1,\bar{0},2)(\bar{1},0,\bar{2})02\bar{1}02\bar{1}.$ ∎

Since $N^{(01\bar{2}01\bar{2})} \geq N^{01\bar{2}01\bar{2}} =< e >$ and
$N(t_0t_1\bar{t_2}t_0t_1\bar{t_2})^{\{(1,\bar{1})(\bar{2},2)\}} = Nt_0\bar{t_1}t_2t_0\bar{t_1}t_2 \Rightarrow (1,\bar{1})(\bar{2},2) \in N^{(01\bar{2}01\bar{2})}$
$N(t_0t_1\bar{t_2}t_0t_1\bar{t_2})^{\{(1,2)(\bar{1},\bar{2})\}} = Nt_0t_2\bar{t_1}t_0t_2\bar{t_1} \Rightarrow (1,2)(\bar{1},\bar{2}) \in N^{(01\bar{2}01\bar{2})}$
$\Rightarrow N^{(01\bar{2}01\bar{2})} =< (1,\bar{1})(\bar{2},2), (1,2)(\bar{1},\bar{2}) >$

$= \{e, (1,\bar{2})(2,\bar{1}), (1,2)(\bar{1},\bar{2}), (1,\bar{1})(2,\bar{2})\}$.

Therefore, the number of single cosets in the double coset $[01\bar{2}01\bar{2}]$ are at most

$\frac{|N|}{|N^{(01\bar{2}01\bar{2})}|} = \frac{4!}{4} = \frac{24}{4} = 6$. Each has four different names.

Now, take $(01\bar{2}01\bar{2})$ conjugates with the elements of $N^{(01\bar{2}01\bar{2})}$, we obtain the following relations:

$$01\bar{2}01\bar{2} \sim 0\bar{1}20\bar{1}2 \sim 02\bar{1}02\bar{1} \sim 0\bar{2}10\bar{2}1$$

By conjugating $(01\bar{2}01\bar{2} \sim 0\bar{1}20\bar{1}2 \sim 02\bar{1}02\bar{1} \sim 0\bar{2}10\bar{2}1)$ with the elements of transversal of $N^{(01\bar{2}01\bar{2})}$, we obtain the following six single cosets of double coset $[01\bar{2}01\bar{2}]$:

$$\begin{gathered}
01\bar{2}01\bar{2} \sim 0\bar{1}20\bar{1}2 \sim 02\bar{1}02\bar{1} \sim 0\bar{2}10\bar{2}1 \\
1\bar{0}21\bar{0}2 \sim 10\bar{2}10\bar{2} \sim 1\bar{2}01\bar{2}0 \sim 12\bar{0}12\bar{0} \\
\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}\bar{2} \sim \bar{0}12\bar{0}12 \sim \bar{0}21\bar{0}21 \sim \bar{0}\bar{2}\bar{1}\bar{0}\bar{2}\bar{1} \\
\bar{2}\bar{0}\bar{1}\bar{2}\bar{0}\bar{1} \sim \bar{2}01\bar{2}01 \sim \bar{2}10\bar{2}10 \sim \bar{2}\bar{1}\bar{0}\bar{2}\bar{1}\bar{0} \\
\bar{1}02\bar{1}02 \sim \bar{1}\bar{0}\bar{2}\bar{1}\bar{0}\bar{2} \sim \bar{1}\bar{2}\bar{0}\bar{1}\bar{2}\bar{0} \sim \bar{1}20\bar{1}20 \\
20\bar{1}20\bar{1} \sim 2\bar{0}12\bar{0}1 \sim 21\bar{0}21\bar{0} \sim 2\bar{1}02\bar{1}0
\end{gathered}$$

The orbits of $N^{(01\bar{2}01\bar{2})}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{\bar{0}\}$ and $\{1,2,\bar{1},\bar{2}\}$. Now, take one representative from each orbit, namely 0, $\bar{0}$ and 2, then find the double coset to which $Nt_0t_1\bar{t_2}t_0t_1\bar{t_2}t_0$, $Nt_0t_1\bar{t_2}t_0t_1\bar{t_2}\bar{t_0}$ and $Nt_0t_1\bar{t_2}t_0t_1\bar{t_2}t_2$ belong? However,

$Nt_0t_1\bar{t_2}t_0t_1\bar{t_2}t_0 \in Nt_0t_1\bar{t_2}t_0t_1\bar{t_2}t_0N$ (New double coset denote by $[01\bar{2}01\bar{2}0]$).
$Nt_0t_1\bar{t_2}t_0t_1\bar{t_2}\bar{t_0} = Nt_0t_1\bar{t_2}t_0t_1\bar{t_2}\bar{t_0} \in [01\bar{2}01\bar{2}]$ **(By Lemma-34).**
$Nt_0t_1\bar{t_2}t_0t_1\bar{t_2}t_2 = Nt_0t_1\bar{t_2}t_0t_1 \in [01\bar{2}01]$.

Consider the new double coset [0120120].

Claim: $0120120 \sim \bar{2}\bar{1}0\bar{2}\bar{1}0\bar{2} \sim \bar{2}0\bar{1}\bar{2}0\bar{1}\bar{2}$

Proof. By using our relation conjugates with elements of N, we have
1. $01201\underline{20} = 01201(2,0)(\bar{2},\bar{0})\bar{2}\bar{0}\bar{2} = (2,0)(\bar{2},\bar{0})210\underline{21}\bar{2}\bar{0}\bar{2}$
$= (2,0)(\bar{2},\bar{0})210(1,2)(\bar{1},\bar{2})\bar{2}\bar{1}\bar{2}\bar{2}\bar{0}\bar{2} = (1,2,0)(\bar{1},\bar{2},\bar{0})120\bar{2}\bar{1}2\bar{0}\bar{2}$
$= (1,2,0)(\bar{1},\bar{2},\bar{0})1202\underline{2\bar{1}2}\bar{0}\bar{2} = (1,2,0)(\bar{1},\bar{2},\bar{0})1202(2,\bar{1})(\bar{2},1)\bar{2}1\bar{0}\bar{2}$

$= (1,\bar{1})(2,0,\bar{2},\bar{0})\bar{2}\underline{\bar{1}0\bar{1}}\bar{2}1\bar{0}\bar{2} = (1,\bar{1})(2,0,\bar{2},\bar{0})\bar{2}(0,\bar{1})(\bar{0},1)1\bar{0}\bar{2}1\bar{0}\bar{2}$
$= (1,0,\bar{2})(\bar{1},\bar{0},2)\bar{2}1\underline{\bar{0}\bar{2}}1\bar{0}\bar{2} = (1,0,\bar{2})(\bar{1},\bar{0},2)\bar{2}1(2,0)(\bar{2},\bar{0})0201\bar{0}\bar{2}$
$= (0,\bar{0})(1,2,\bar{1},\bar{2})\bar{0}102\underline{01}\bar{0}\bar{2} = (0,\bar{0})(1,2,\bar{1},\bar{2})\bar{0}102(1,0)(\bar{1},\bar{0})\bar{0}\bar{1}\bar{0}0\bar{2}$
$= (1,2,\bar{0})(\bar{1},\bar{2},0)\bar{1}01\underline{2\bar{0}}\bar{1}0\bar{2} = (1,2,\bar{0})(\bar{1},\bar{2},0)\bar{1}01(2,\bar{0})(\bar{2},0)\bar{2}0\bar{2}\bar{1}0\bar{2}$
$= (1,\bar{0})(\bar{1},0)\bar{1}\underline{\bar{2}1\bar{2}}0\bar{2}\bar{1}0\bar{2} = (1,\bar{0})(\bar{1},0)\bar{1}(1,\bar{2})(\bar{1},2)2\bar{1}0\bar{2}\bar{1}0\bar{2}$
$= (1,\bar{0},\bar{2})(\bar{1},0,2)22\bar{1}0\bar{2}\bar{1}0\bar{2} = (1,\bar{0},\bar{2})(\bar{1},0,2)\bar{2}\bar{1}0\bar{2}\bar{1}0\bar{2}.$

2. $012\underline{01}20 = 012(1,0)(\bar{1},\bar{0})\bar{0}\bar{1}\bar{0}20 = (1,0)(\bar{1},\bar{0})102\bar{0}\bar{1}\bar{0}20$
$= (1,0)(\bar{1},\bar{0})1\underline{020}0\bar{1}\bar{0}20 = (1,0)(\bar{1},\bar{0})1(2,0)(\bar{2},\bar{0})\bar{0}\bar{2}0\bar{1}\bar{0}20$
$= (1,2,0)(\bar{1},\bar{2},\bar{0})1\bar{0}\bar{2}\underline{0\bar{1}}\bar{0}20 = (1,2,0)(\bar{1},\bar{2},\bar{0})1\bar{0}\bar{2}(1,\bar{0})(\bar{1},0)\bar{0}1\bar{0}\bar{0}20$
$= (0,\bar{0})(1,2,\bar{1},\bar{2})\bar{0}1\bar{2}\bar{0}1\underline{020} = (0,\bar{0})(1,2,\bar{1},\bar{2})\bar{0}1\bar{2}\bar{0}1(2,0)(\bar{2},\bar{0})\bar{0}\bar{2}$
$= (1,0,\bar{2})(\bar{1},\bar{0},2)\underline{\bar{2}1}\bar{0}\bar{2}1\bar{0}\bar{2} = (1,0,\bar{2})(\bar{1},\bar{0},2)(1,\bar{2})(\bar{1},2)2\bar{1}2\bar{0}\bar{2}1\bar{0}\bar{2}$
$= (1,0)(\bar{1},\bar{0})2\bar{1}2\bar{0}\bar{2}1\bar{0}\bar{2} = (1,0)(\bar{1},\bar{0})2\bar{1}\bar{2}\underline{2\bar{0}\bar{2}}1\bar{0}\bar{2}$
$= (1,0)(\bar{1},\bar{0})2\bar{1}\bar{2}(2,0)(\bar{2},\bar{0})201\bar{0}\bar{2} = (1,2,0)(\bar{1},\bar{2},\bar{0})0\bar{1}\bar{0}2\underline{01}\bar{0}\bar{2}$
$= (1,2,0)(\bar{1},\bar{2},\bar{0})0\bar{1}\bar{0}2(1,0)(\bar{1},\bar{0})\bar{0}\bar{1}\bar{0}\bar{0}\bar{2} = (1,2)(\bar{1},\bar{2})1\bar{0}\bar{1}2\bar{0}\bar{1}0\bar{2}$
$= (1,2)(\bar{1},\bar{2})1\bar{0}\bar{1}2\underline{\bar{0}\bar{1}\bar{0}}0\bar{2} = (1,2)(\bar{1},\bar{2})1\bar{0}\bar{1}2(1,0)(\bar{1},\bar{0})01\bar{0}\bar{2}$
$= (1,2,0)(\bar{1},\bar{2},\bar{0})0\bar{1}\underline{\bar{0}2}01\bar{0}\bar{2} = (1,2,0)(\bar{1},\bar{2},\bar{0})0\bar{1}(\bar{0},2)(0,\bar{2})0\bar{2}001\bar{0}\bar{2}$
$= (2,\bar{2})(1,\bar{0},\bar{1},0)\bar{2}\bar{1}0\bar{2}\underline{\bar{0}1\bar{0}}\bar{2} = (2,\bar{2})(1,\bar{0},\bar{1},0)\bar{2}\bar{1}0\bar{2}(1,\bar{0})(\bar{1},0)0\bar{1}\bar{2}$
$= (0,\bar{0})(\bar{2},2)\bar{2}0\bar{1}\bar{2}0\bar{1}\bar{2}.$ □

Since $N^{(0120120)} \geq N^{0120120} =< e >$ and
$N(t_0t_1t_2t_0t_1t_2t_0)^{\{(1,\bar{1})(2,0,\bar{2},\bar{0})\}} = N\bar{t_2}\bar{t_1}t_0\bar{t_2}\bar{t_1}t_0\bar{t_2} \Rightarrow (1,\bar{1})(2,0,\bar{2},\bar{0}) \in N^{(0120120)}$.
$N(t_0t_1t_2t_0t_1t_2t_0)^{\{(1,0,\bar{2})(2,\bar{1},\bar{0})\}} = N\bar{t_2}t_0\bar{t_1}\bar{t_2}t_0\bar{t_1}\bar{t_2} \Rightarrow (1,0,\bar{2})(2,\bar{1},\bar{0}) \in N^{(0120120)}$.
$\Rightarrow N^{(0120120)} =< (1,\bar{1})(2,0,\bar{2},\bar{0}), (1,0,\bar{2})(2,\bar{1},\bar{0}) > \cong S_4$.

Therefore, the number of single cosets in the double coset [0120120] are at most

$\frac{|N|}{|N^{(0120120)}|} = \frac{4!}{4!} = \frac{24}{24} = 1$. This single coset has twenty four different names.

Now, take (0120120) conjugates with the elements of $N^{(0120120)} \cong S_4$, we obtain the following relations:
$0120120 \sim \bar{2}\bar{1}0\bar{2}0\bar{2} \sim 1\bar{2}\bar{0}1\bar{2}\bar{0}1 \sim 0\bar{1}\bar{2}0\bar{1}\bar{2}0 \sim \bar{0}2\bar{1}\bar{0}2\bar{1}\bar{0} \sim \bar{1}\bar{0}2\bar{1}\bar{0}2\bar{1} \sim \bar{2}0\bar{1}\bar{2}0\bar{1}\bar{2} \sim 0\bar{2}\bar{1}0\bar{2}\bar{1}0 \sim$
$\bar{2}1\bar{0}\bar{2}1\bar{0}\bar{2} \sim \bar{1}2\bar{0}\bar{1}2\bar{0}\bar{1} \sim 2\bar{0}\bar{1}2\bar{0}\bar{1}2 \sim 2102102 \sim 1\bar{0}\bar{2}1\bar{0}\bar{2}1 \sim \bar{1}0\bar{2}\bar{1}0\bar{2}\bar{1} \sim 1021021 \sim \bar{0}1\bar{2}\bar{0}1\bar{2}\bar{0} \sim$

$\bar{1}20\bar{1}20\bar{1} \sim \bar{2}01\bar{2}01\bar{2} \sim 2012012 \sim 0210210 \sim \bar{0}\bar{2}1\bar{0}\bar{2}1\bar{0} \sim \bar{0}\bar{1}2\bar{0}\bar{1}2\bar{0} \sim 2\bar{1}\bar{0}2\bar{1}\bar{0}2 \sim 1201201.$

Since $N^{(0120120)}$ acts transitively on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$. Thus, $N^{(0120120)}$ has only single orbit $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$.
Now take one representative $\bar{0}$ from the orbit and determine which double coset does $Nt_0t_1t_2t_0t_1t_2t_0\bar{t_0}$ belong? However,
$Nt_0t_1t_2t_0t_1t_2t_0\bar{t_0} = Nt_0t_1t_2t_0t_1t_2 \in [012012]$.

Consider the new double coset $[01\bar{2}01\bar{2}0]$.

Claim: $01\bar{2}01\bar{2}0 \sim \bar{2}\bar{1}\bar{0}\bar{2}\bar{1}\bar{0}\bar{2} \sim \bar{0}21\bar{0}21\bar{0}$

Proof. By using our relation conjugates with elements of N, we have

1. $01\bar{2}01\bar{2}0 = 01\bar{2}012\underline{20} = 01\bar{2}012(2,0)(\bar{2},\bar{0})\bar{2}\bar{0}\bar{2} = (2,0)(\bar{2},\bar{0})21\bar{0}2\underline{10}\bar{2}\bar{0}\bar{2}$
$= (2,0)(\bar{2},\bar{0})21\bar{0}2(1,0)(\bar{1},\bar{0})\bar{1}\bar{0}\bar{1}\bar{2}\bar{0}\bar{2} = (2,1,0)(\bar{2},\bar{1},\bar{0})20\bar{1}2\bar{1}\bar{0}\underline{\bar{1}\bar{2}}\bar{0}\bar{2}$
$= (2,1,0)(\bar{2},\bar{1},\bar{0})20\bar{1}2\bar{1}\bar{0}(2,1)(\bar{2},\bar{1})121\bar{0}\bar{2} = (1,0)(\bar{1},\bar{0})10\underline{\bar{2}1\bar{2}}\bar{0}121\bar{0}\bar{2}$
$= (1,0)(\bar{1},\bar{0})10(1,\bar{2})(\bar{1},2)2\bar{1}\bar{0}121\bar{0}\bar{2} = (0,\bar{2},1)(\bar{0},2,\bar{1})\bar{2}02\bar{1}\bar{0}121\bar{0}\bar{2}$
$= (0,\bar{2},1)(\bar{0},2,\bar{1})\bar{2}02\underline{\bar{1}\bar{0}\bar{1}}\bar{1}21\bar{0}\bar{2} = (0,\bar{2},1)(\bar{0},2,\bar{1})\bar{2}02(1,0)(\bar{1},\bar{0})10\bar{1}21\bar{0}\bar{2}$
$= (0,\bar{2})(\bar{0},2)\bar{2}1210\underline{\bar{1}2}1\bar{0}\bar{2} = (0,\bar{2})(\bar{0},2)\bar{2}1210(1,\bar{2})(\bar{1},2)1\bar{2}11\bar{0}\bar{2}$
$= (0,1,\bar{2})(\bar{0},\bar{1},2)1\underline{\bar{2}1\bar{2}}01\bar{2}\bar{1}\bar{0}\bar{2} = (0,1,\bar{2})(\bar{0},\bar{1},2)1(1,2)(\bar{1},\bar{2})2101\bar{2}\bar{1}\bar{0}\bar{2}$
$= (1,\bar{1})(0,2,\bar{0},\bar{2})22101\bar{2}\bar{1}\bar{0}\bar{2} = (1,\bar{1})(0,2,\bar{0},\bar{2})\bar{2}\underline{101}\bar{2}\bar{1}\bar{0}\bar{2}$
$= (1,\bar{1})(0,2,\bar{0},\bar{2})\bar{2}(1,0)(\bar{1},\bar{0})\bar{1}\bar{0}\bar{2}\bar{1}\bar{0}\bar{2} = (0,2,\bar{1})(\bar{0},\bar{2},1)\bar{2}\bar{1}\bar{0}\bar{2}\bar{1}\bar{0}\bar{2}.$

2. $01\bar{2}0\underline{1\bar{2}}0 = 01\bar{2}0(1,\bar{2})(\bar{1},2)\bar{1}2\bar{1}0 = (1,\bar{2})(\bar{1},2)0\bar{2}10\bar{1}2\bar{1}0 = (1,\bar{2})(\bar{1},2)0\bar{2}10\bar{1}2\underline{\bar{1}\bar{0}}\bar{0}$
$= (1,\bar{2})(\bar{1},2)0\bar{2}10\bar{1}2(1,0)(\bar{1},\bar{0})101\bar{0} = (1,\bar{2},0)(\bar{1},2,\bar{0})1\bar{2}01\bar{0}\underline{21}01\bar{0}$
$= (1,\bar{2},0)(\bar{1},2,\bar{0})1\bar{2}01\bar{0}(1,2)(\bar{1},\bar{2})\bar{2}\bar{1}\bar{2}01\bar{0} = (1,\bar{1})(0,2,\bar{0},\bar{2})2\bar{1}02\bar{0}\bar{2}\bar{1}\underline{\bar{2}0}1\bar{0}$
$= (1,\bar{1})(0,2,\bar{0},\bar{2})2\bar{1}02\bar{0}\bar{2}\bar{1}(0,\bar{2})(\bar{0},2)2\bar{0}21\bar{0} = (1,\bar{1})(\bar{0},0)\bar{0}\bar{1}\underline{\bar{2}\bar{0}}20\bar{1}2\bar{0}21\bar{0}$
$= (1,\bar{1})(\bar{0},0)\bar{0}\bar{1}(2,0)(\bar{2},\bar{0})20220\bar{1}2\bar{0}21\bar{0} = (1,\bar{1})(0,\bar{2},\bar{0},2)\underline{\bar{2}\bar{1}}20\bar{2}0\bar{1}2\bar{0}21\bar{0}$
$= (1,\bar{1})(0,\bar{2},\bar{0},2)(2,1)(\bar{2},\bar{1})21220\bar{2}0\bar{1}2\bar{0}21\bar{0} = (1,\bar{2},\bar{0})(\bar{1},2,0)21\underline{\bar{2}0\bar{2}}0\bar{1}2\bar{0}21\bar{0}$
$= (1,\bar{2},\bar{0})(\bar{1},2,0)21(0,\bar{2})(\bar{0},2)2\bar{0}0\bar{1}2\bar{0}21\bar{0} = (2,\bar{2})(1,0,\bar{1},\bar{0})\bar{0}1\underline{\bar{2}\bar{1}\bar{2}}\bar{0}21\bar{0}$
$= (2,\bar{2})(1,0,\bar{1},\bar{0})\bar{0}1(1,\bar{2})(\bar{1},2)\bar{2}1\bar{0}21\bar{0} = (1,0,2)(\bar{1},\bar{0},\bar{2})\bar{0}\bar{2}\bar{2}1\bar{0}21\bar{0}$
$= (1,0,2)(\bar{1},\bar{0},\bar{2})\bar{0}21\bar{0}21\bar{0}.$ $\square$

Since $N^{(01\bar{2}01\bar{2}0)} \geq N^{01\bar{2}01\bar{2}0} =< e >$ and
$N(t_0t_1\bar{t_2}t_0t_1\bar{t_2}(t_0)^{\{(1,\bar{1})(2,0,\bar{2},\bar{0})\}} = N\bar{t_2}\bar{t_1}\bar{t_0}\bar{t_2}\bar{t_1}\bar{t_0}\bar{t_2} \Rightarrow (1,\bar{1})(2,0,\bar{2},\bar{0}) \in N^{(01\bar{2}01\bar{2}0)}$.
$N(t_0t_1\bar{t_2}t_0t_1\bar{t_2}(t_0)^{\{(0,\bar{0})(1,2,\bar{1},\bar{2})\}} = N\bar{t_0}t_2t_1\bar{t_0}t_2t_1\bar{t_0} \Rightarrow (0,\bar{0})(1,2,\bar{1},\bar{2}) \in N^{(01\bar{2}01\bar{2}0)}$.
$\Rightarrow N^{(01\bar{2}01\bar{2})} =< (1,\bar{1})(2,0,\bar{2},\bar{0}),(0,\bar{0})(1,2,\bar{1},\bar{2}) > \cong S_4$.

Therefore, the number of single cosets in the double coset $[01\bar{2}01\bar{2}0]$ are at most

$\frac{|N|}{|N^{(01\bar{2}01\bar{2}0)}|} = \frac{4!}{4!} = \frac{24}{24} = 1$. This single coset has twenty four different names.

Now, take $(01\bar{2}01\bar{2}0)$ conjugates with the elements of $N^{(01\bar{2}01\bar{2}0)} \cong S_4$, we obtain the following relations:
$01\bar{2}01\bar{2}0 \sim \bar{1}0\bar{2}\bar{1}0\bar{2}\bar{1} \sim \bar{0}12\bar{0}12\bar{0} \sim \bar{2}\bar{1}0\bar{2}\bar{1}0\bar{2} \sim \bar{1}20\bar{1}20\bar{1} \sim \bar{1}\bar{2}\bar{0}\bar{1}\bar{2}\bar{0}\bar{1} \sim \bar{0}\bar{2}\bar{1}\bar{0}\bar{2}\bar{1}\bar{0} \sim 0\bar{2}10\bar{2}10 \sim$
$21\bar{0}21\bar{0}2 \sim \bar{2}01\bar{2}01\bar{2} \sim \bar{2}10\bar{2}10\bar{2} \sim 2\bar{1}02\bar{1}02 \sim 0\bar{1}20\bar{1}20 \sim \bar{0}21\bar{0}21\bar{0} \sim \bar{0}\bar{1}\bar{2}\bar{0}\bar{1}\bar{2}\bar{0} \sim 2\bar{0}12\bar{0}12 \sim$
$02\bar{1}02\bar{1}0 \sim 12\bar{0}12\bar{0}1 \sim 1\bar{2}01\bar{2}01 \sim \bar{1}02\bar{1}02\bar{1} \sim 10\bar{2}10\bar{2}1 \sim 1\bar{2}21\bar{2}21 \sim \bar{2}\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}\bar{2} \sim 20\bar{1}20\bar{1}2$.

Since $N^{(01\bar{2}01\bar{2}0)}$ acts transitively on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$. Thus, $N^{(01\bar{2}01\bar{2}0)}$ has only single orbit $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$.
Now take one representative $\bar{0}$from the orbit and determine which double coset does $Nt_0t_1\bar{t_2}t_0t_1\bar{t_2}t_0\bar{t_0}$ belong? However,
$Nt_0t_1\bar{t_2}t_0t_1\bar{t_2}t_0\bar{t_0} = Nt_0t_1\bar{t_2}t_0t_1\bar{t_2} \in [01\bar{2}01\bar{2}]$.

Since all symmetric generators go back to the previous double cosets and none extend. Thus, we must have completed the double coset enumeration. All the information is summarized in the Cayley diagram below.

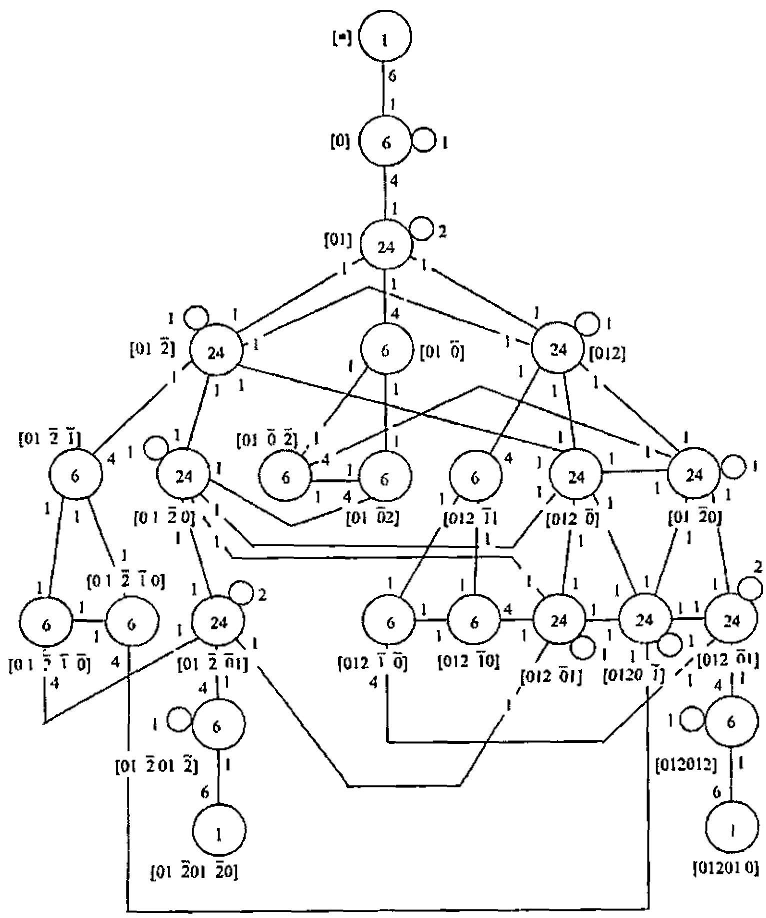

Figure 6.1: The Cayley diagram of $3^{\cdot}A_7$ over S_4.

The double coset enumeration shows that the index of $N \cong S_4$ is at most:
$|G| \leq (315)|N| = (315) \times 24 = 7560$.

The homomorphic image of G.

Define $\widehat{\alpha} : G \to S_{315}$, where (315) is a number of single cosets from the Cayley diagram above gives the action of G on the 315 cosets. From the permutation table, we compute the action of x, y and t_0 on the 315 cosets as follow:
$\widehat{\alpha}(x) = (2,4,3,5)(6,9)(7,14,11,15)...(306,310,307,311)(308,312,309,313)$.
$\widehat{\alpha}(y) = (4,6)(5,9)(7,12)(8,13)(10,18)(11,19)...(299,307)(301,308)(305,309)$.
$\widehat{\alpha}(t_0) = (1,2,3)(4,7,8)(5,10,11)...(310,314,311)(312,315,313)$.
In order to verify that $\widehat{\alpha}$ is a homomorphism, we required

1. We want to show that $\widehat{\alpha}(t_0)$ has exactly 6 conjugates under conjugation by $< \widehat{\alpha}(x), \widehat{\alpha}(y) >$.
   $\widehat{\alpha}(t_0)^{\widehat{\alpha}(x)} = (1,4,5)(2,17,15)(3,14,16)...(306,307,314)(308,309,315) = \widehat{\alpha}(t_1)$.
   $\widehat{\alpha}(t_1)^{\widehat{\alpha}(x)} = (1,3,2)(4,8,7)(5,11,10)...(310,311,314)(312,313,315) = \widehat{\alpha}(\bar{t_0})$.
   $\widehat{\alpha}(\bar{t_0})^{\widehat{\alpha}(x)} = (1,5,4)(2,15,17)(3,16,14)...(306,314,307)(308,315,309) = \widehat{\alpha}(\bar{t_1})$.
   $\widehat{\alpha}(\bar{t_1})^{\widehat{\alpha}(x)} = (1,2,3)(4,7,8)(5,10,11)...(310,314,311)(312,315,313) = \widehat{\alpha}(t_0)$.
   $\widehat{\alpha}(t_2)^{\widehat{\alpha}(x)} = (1,6,9)(2,27,24)(3,22,25)...(295,299,314)(301,305,315) = \widehat{\alpha}(\bar{t_2})$.
   $\widehat{\alpha}(\bar{t_2})^{\widehat{\alpha}(x)} = (1,9,6)(2,24,27)(3,25,22)...(295,314,299)(301,315,305) = \widehat{\alpha}(t_2)$.

   $\widehat{\alpha}(t_1)^{\widehat{\alpha}(y)} = (1,6,9)(2,27,24)(3,22,25)...(295,299,314)(301,305,315) = \widehat{\alpha}(\bar{t_2})$.
   $\widehat{\alpha}(\bar{t_2})^{\widehat{\alpha}(y)} = (1,4,5)(2,17,15)(3,14,16)...(306,307,314)(308,309,315) = \widehat{\alpha}(t_1)$.
   $\widehat{\alpha}(t_2)^{\widehat{\alpha}(y)} = (1,5,4)(2,15,17)(3,16,14)...(306,314,307)(308,315,309) = \widehat{\alpha}(\bar{t_1})$.
   $\widehat{\alpha}(\bar{t_1})^{\widehat{\alpha}(y)} = (1,9,6)(2,24,27)(3,25,22)...(295,314,299)(301,315,305) = \widehat{\alpha}(t_2)$.

2. $\widehat{\alpha}(N)$ acts as N on $\{\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(t_2), \widehat{\alpha}(\bar{t_0}), \widehat{\alpha}(\bar{t_1}), \widehat{\alpha}(\bar{t_2})\}$ by conjugation, that is, if
   $x = (0,1,\bar{0},\bar{1})(2,\bar{2})$, and
   $y = (1,\bar{2})(2,\bar{1})$, then
   $\widehat{\alpha}(x) = \{(\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(\bar{t_0}), \widehat{\alpha}(\bar{t_1}))(\widehat{\alpha}(t_2), \widehat{\alpha}(\bar{t_2})\}$,

$\widehat{\alpha}(y) = \{(\widehat{\alpha}(t_1), \widehat{\alpha}(\bar{t_2}))(\widehat{\alpha}(t_1), \widehat{\alpha}(t_2))\}$.

Then, $\widehat{\alpha}(G) = \widehat{\alpha}(3^{*3} : S_4) = \widehat{\alpha} < N, t_0 > = < \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$ is a homomorphic image of $(3^{*3} : S_4)$.

Next, we want the homomorphic image of $G = \frac{3^{*3}:S_4}{[(1,\bar{2})(\bar{1},2)t_1]^5}$.

Since, $< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) > \leq S_{315}$ is a homomorphic image of $G = \frac{3^{*3}:S_4}{[(1,\bar{2})(\bar{1},2)t_1]^5}$ if the additional relation

$(1, \bar{2})(\bar{1}, 2)t_1\bar{t_2}t_1 = \bar{t_1}t_2$ holds in $< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$.

Now, we want to show that $\widehat{\alpha}((1, \bar{2})(\bar{1}, 2)t_1\bar{t_2}t_1) = \widehat{\alpha}(\bar{t_1}t_2)$.

On the left side of our equation

$\widehat{\alpha}((1, \bar{2})(\bar{1}, 2)t_1\bar{t_2}t_1) = \widehat{\alpha}((1, \bar{2})(\bar{1}, 2))\widehat{\alpha}(t_1)\widehat{\alpha}(\bar{t_2})\widehat{\alpha}(t_1)$

$= (4, 6)(5, 9)(7, 12)(8, 13)(10, 18)(11, 19)...(299, 307)(301, 308)(305, 309)$

$(1, 4, 5)(2, 17, 15)(3, 14, 16)...(306, 307, 314)(308, 309, 315)$

$(1, 6, 9)(2, 27, 24)(3, 22, 25)...(295, 299, 314)(301, 305, 315)$

$(1, 4, 5)(2, 17, 15)(3, 14, 16)...(306, 307, 314)(308, 309, 315)$

$= (1, 33, 4, 9, 28)(2, 92, 17, 24, 72)...(224, 278, 255, 262, 259)(228, 290, 234, 266, 237)$.

On the right side of our equation

$\widehat{\alpha}(\bar{t_1}t_2) = \widehat{\alpha}(\bar{t_1})\widehat{\alpha}(t_2)$

$= (1, 5, 4)(2, 15, 17)(3, 16, 14)...(306, 314, 307)(308, 315, 309)$

$(1, 9, 6)(2, 24, 27)(3, 25, 22)...(295, 314, 299)(301, 315, 305)$

$= (1, 33, 4, 9, 28)(2, 92, 17, 24, 72)...(224, 278, 255, 262, 259)(228, 290, 234, 266, 237)$.

Since, $\widehat{\alpha} : G \to S_{315}$ is a homomorphism map. By First Isomorphism Theorem, we have

$G|ker(\widehat{\alpha}) \cong im(\widehat{\alpha})$

$\Rightarrow \frac{|G|}{|ker(\widehat{\alpha})|} = |im(\widehat{\alpha})| = | < \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0)| = 7,560$

$\Rightarrow |G| = 7,560 \times |ker(\widehat{\alpha})|$, where $(|ker(\widehat{\alpha})| \geq 1)$.

$\Rightarrow |G| \geq 7,560$.

But early we found that $|G| \leq 7,560$. Therefore, we can conclude that $|G| = 7,560$. Since $|G| = 7,560$,

$\Rightarrow |ker(\widehat{\alpha})| = 1$.

$\Rightarrow \widehat{\alpha}$ is also an isomorphism.

$\Rightarrow G \cong \widehat{\alpha}(G) = < \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$.

It is clear from the Cayley diagram of G over S_4 that every element of G can be written as πw, where π is a permutation of S_4 on six letters, and w is a word in the t_{i_s} of the length at most seven. We call this representation of elements of G the symmetric representation. We now give examples of elements of G in both symmetric and permutation representation.

Permutation Representation to Symmetric Presentation

Let $p = (1,15)(2,26)(3,8)(4,16)(6,47)(7,14)(9,59)(12,70)(13,41)(17,23)(18,84)$
$(19,32)(20,60)(22,121)(24,105)(25,143)(27,88)...(266,288)(267,313)(269,279)$
$(273,295)(274,314)(275,306)(281,305)(284,296)(289,300)(292,308)(293,315)$
$(294,310)(302,312)$.
be an element in permutation representation.
We know that $Np = 1^p$
$\Rightarrow Np = p(1) = 15$ and the labeling table gives that $Nt_0\bar{t_1}$ is labeled 15.
In order the represent p in the symmetric representation form, we must find $n \in N \cong S_4$.
We have
$Np = 1^p = p(1) = Nt_0\bar{t_1}$. (An element that labeling (15)).
$p = nw$
$\Rightarrow n = pw^{-1}$
$\Rightarrow n = pt_1\bar{t_0}$
$\Rightarrow \widehat{\alpha}(n) = p\widehat{\alpha}(t_1)\widehat{\alpha}(\bar{t_0})$
$= (2,4)(3,5)(7,17)(8,15)(10,14)(11,16)(12,28)(13,29)(18,20)(19,21)(22,30)...$
$(284,298)(285,296)(288,300)(289,303)(292,304)(293,302)(306,311)(307,310)$
$(308,313)(309,312)$
From our multiplication table we have,
$n = (1,0)(\bar{1},\bar{0})$.
Therefore, p can be written in the symmetric representation as
$p = (1,0)(\bar{1},\bar{0})t_0\bar{t_1}$.

Symmetric Presentation to Permutation Representation

Let $\{(1,\bar{2})(\bar{1},2)t_1\bar{t_2}t_1\}$ be an element in the symmetric representation.
$\Rightarrow p = \widehat{\alpha}((1,\bar{2})(\bar{1},2))\widehat{\alpha}(t_1)\widehat{\alpha}(\bar{t_2})\widehat{\alpha}(t_1)$

$= (4,6)(5,9)(7,12)(8,13)(10,18)(11,19)...(299,307)(301,308)(305,309)$
$(1,4,5)(2,17,15)(3,14,16)...(306,307,314)(308,309,315)$
$(1,6,9)(2,27,24)(3,22,25)...(295,299,314)(301,305,315)$
$(1,4,5)(2,17,15)(3,14,16)...(306,307,314)(308,309,315)$
$= (1,33,4,9,28)(2,92,17,24,72)(3,89,14,25,75)(5,37,20,6,53)(7,148,49,39,160)$
$(8,147,46,36,159)(10,69,61,13,156)(11,68,58,12,163)(15,99,62,27,131)$
$(16,103,65,22,136)(18,212,23,82,115)(19,185,26,83,112)(21,29,52,30,38)$
$(31,264,94,117,206)(32,268,90,116,199)(34,165,109,45,242)(35,155,106,44,252)$
$(40,267,126,101,291)(41,263,120,98,279)(42,128,144,57,276)(43,119,142,56,281)$
$(47,157,150,71,193)(48,162,169,66,202)(50,302,81,179,226)(51,296,84,181,222)$
$(54,256,67,139,225)(55,231,70,134,229)(59,192,166,85,151)(60,205,153,80,168)$
$(63,73,130,95,100)(64,74,137,86,102)(76,214,91,135,196)(77,191,93,138,209)$
$(78,293,107,129,248)(79,275,108,118,244)(87,217,189,132,176)(88,243,186,164,239)$
$(96,253,219,154,250)(97,188,216,141,171)(104,289,260,124,312)$
$(105,297,258,180,271)(110,303,236,178,287)(111,285,232,123,311)$
$(113,161,182,149,152)(114,158,213,146,167)(121,277,270,125,298)$
$(122,304,127,283,286)(133,184,143,240,280)(140,220,145,247,273)$
$(170,183,187,241,238)(172,292,197,215,249)(173,307,198,230,295)$
$(174,309,207,257,301)(175,274,208,190,245)(177,221,218,246,251)$
$(194,288,261,210,313)(195,310,211,284,233)(200,300,269,282,227)$
$(201,306,299,235,314)(203,308,305,254,315)(204,294,265,272,223)$
$(224,278,255,262,259)(228,290,234,266,237).$

Therefore, $\{(1,\bar{2})(\bar{1},2)t_1\bar{t_2}t_1\}$ can be written in the permutation representation as follows
$(1,\bar{2})(\bar{1},2)t_1\bar{t_2}t_1$
$= (1,33,4,9,28)(2,92,17,24,72)(3,89,14,25,75)(5,37,20,6,53)(7,148,49,39,160)$
$(8,147,46,36,159)(10,69,61,13,156)(11,68,58,12,163)(15,99,62,27,131)$
$(16,103,65,22,136)(18,212,23,82,115)(19,185,26,83,112)(21,29,52,30,38)$
$(31,264,94,117,206)(32,268,90,116,199)(34,165,109,45,242)(35,155,106,44,252)$
$(40,267,126,101,291)(41,263,120,98,279)(42,128,144,57,276)(43,119,142,56,281)$
$(47,157,150,71,193)(48,162,169,66,202)(50,302,81,179,226)(51,296,84,181,222)$
$(54,256,67,139,225)(55,231,70,134,229)(59,192,166,85,151)(60,205,153,80,168)$

(63, 73, 130, 95, 100)(64, 74, 137, 86, 102)(76, 214, 91, 135, 196)(77, 191, 93, 138, 209)
(78, 293, 107, 129, 248)(79, 275, 108, 118, 244)(87, 217, 189, 132, 176)(88, 243, 186, 164, 239)
(96, 253, 219, 154, 250)(97, 188, 216, 141, 171)(104, 289, 260, 124, 312)
(105, 297, 258, 180, 271)(110, 303, 236, 178, 287)(111, 285, 232, 123, 311)
(113, 161, 182, 149, 152)(114, 158, 213, 146, 167)(121, 277, 270, 125, 298)
(122, 304, 127, 283, 286)(133, 184, 143, 240, 280)(140, 220, 145, 247, 273)
(170, 183, 187, 241, 238)(172, 292, 197, 215, 249)(173, 307, 198, 230, 295)
(174, 309, 207, 257, 301)(175, 274, 208, 190, 245)(177, 221, 218, 246, 251)
(194, 288, 261, 210, 313)(195, 310, 211, 284, 233)(200, 300, 269, 282, 227)
(201, 306, 299, 235, 314)(203, 308, 305, 254, 315)(204, 294, 265, 272, 223)
(224, 278, 255, 262, 259)(228, 290, 234, 266, 237).

6.2 Lemmas

Lemma-1

Proof. $010 = (0,1)(\bar{0},\bar{1})\bar{0}\bar{1} \in [01]$. □

Lemma-2

Proof. $0\underline{121} = 0(2,1)(\bar{2},\bar{1})\bar{1}\bar{2} = (2,1)(\bar{2},\bar{1})0\bar{1}\bar{2} \in [012]$. □

Lemma-3

Proof. $\underline{01}\bar{2}\bar{0} = (0,1)(\bar{0},\bar{1})\bar{0}\bar{1}\underline{\bar{0}\bar{2}\bar{0}} = (0,1)(\bar{0},\bar{1})\bar{0}\bar{1}(2,0)(\bar{2},\bar{0})02$
$= (0,1,2)(\bar{0},\bar{1},\bar{2})\bar{2}\bar{1}02 \in [012\bar{0}]$. □

Lemma-4

Proof. $01\underline{202} = 01(2,0)(\bar{2},\bar{0})\bar{2}\bar{0} = (2,0)(\bar{2},\bar{0})21\bar{2}\bar{0} \in [01\bar{0}\bar{2}]$. □

Lemma-5

Proof. $0120\bar{2} = 01\underline{202}2 = 01(0,2)(\bar{0},\bar{2})\bar{2}\bar{0}2 = (0,2)(\bar{0},\bar{2})\underline{21}\bar{2}\bar{0}2$
$= (0,2)(\bar{0},\bar{2})(1,2)(\bar{1},\bar{2})\bar{2}\bar{1}\bar{2}\bar{2}\bar{0}\bar{2} = (0,1,2)(\bar{0},\bar{1},\bar{2})\bar{2}\bar{1}\underline{\bar{2}\bar{0}\bar{2}}$
$= (0,1,2)(\bar{0},\bar{1},\bar{2})\bar{2}\bar{1}(2,\bar{0})(\bar{2},0)\bar{2}0 = (2,\bar{2})(0,1,\bar{0},\bar{1})0\bar{1}\bar{2}0 \in [0120]$. □

Lemma-6

Proof. $01\underline{2\bar{0}2} = 01(2,\bar{0})(\bar{2},0)\bar{2}0 = (2,\bar{0})(\bar{2},0)\underline{\bar{2}1\bar{2}}0 = (2,\bar{0})(\bar{2},0)(2,\bar{1})(\bar{2},1)2\bar{1}0$
$= (1,\bar{2},0)(2,\bar{0},\bar{1})2\bar{1}0 \in [01\bar{2}]$. ☐

Lemma-7

Proof. $0\underline{12}\bar{0}\bar{1} = 0(1,2)(\bar{1},\bar{2})\bar{1}\bar{2}\bar{1}\bar{0}\bar{1} = (1,2)(\bar{1},\bar{2})0\bar{1}\bar{2}\underline{\bar{1}\bar{0}\bar{1}} = (1,2)(\bar{1},\bar{2})0\bar{1}\bar{2}(1,0)(\bar{1},\bar{0})10$
$= (0,1,2)(\bar{0},\bar{1},\bar{2})1\bar{0}\bar{2}10 \in [0120\bar{1}]$. ☐

Lemma-8

Proof. $\underline{01}2\bar{0}\bar{2} = (1,0)(\bar{1},\bar{0})\bar{0}\bar{1}\underline{\bar{0}2\bar{0}}\bar{2} = (1,0)(\bar{1},\bar{0})\bar{0}\bar{1}(2,\bar{0})(\bar{2},0)0\bar{2}\bar{2}$
$= (0,1,\bar{2})(\bar{0},\bar{1},2)2\bar{1}02 \in [01\bar{2}0]$. ☐

Lemma-9

Proof. $\underline{01}\bar{0}20 = (1,0)(\bar{1},\bar{0})\bar{0}\bar{1}\bar{0}020 = (1,0)(\bar{1},\bar{0})\bar{0}\bar{1}\underline{020} = (1,0)(\bar{1},\bar{0})\bar{0}\bar{1}(2,0)(\bar{2},\bar{0})\bar{0}\bar{2}$
$= (0,1,2)(\bar{0},\bar{1},\bar{2})\bar{2}\bar{1}\bar{0}\bar{2} \in [01\bar{2}0]$. ☐

Lemma-10

Proof. $01\bar{0}\bar{2}0 = \underline{010}0\bar{2}0 = (1,0)(\bar{1},\bar{0})\bar{0}\bar{1}\underline{0\bar{2}0} = (1,0)(\bar{1},\bar{0})\bar{0}\bar{1}(2,\bar{0})(\bar{2},0)\bar{0}2$
$= (0,1,\bar{2})(\bar{0},\bar{1},2)2\bar{1}\bar{0}2 \in [0120]$. ☐

Lemma-11

Proof. $01\bar{2}02 = 012\underline{202} = 012(2,0)(\bar{2},\bar{0})\bar{2}\bar{0} = (2,0)(\bar{2},\bar{0})\underline{21}0\bar{2}\bar{0}$
$= (2,0)(\bar{2},\bar{0})(2,1)(\bar{2},\bar{1})\bar{2}\bar{1}\bar{2}0\bar{2}\bar{0} = (0,1,2)(\bar{0},\bar{1},\bar{2})\bar{2}\bar{1}\underline{\bar{2}0\bar{2}}\bar{0}$
$= (0,1,2)(\bar{0},\bar{1},\bar{2})\bar{2}\bar{1}(2,\bar{0})(\bar{2},0)2\bar{0}\bar{0} = (2,\bar{2})(0,1,\bar{0},\bar{1})0\bar{1}20 \in [01\bar{2}0]$. ☐

Lemma-12

Proof. $0\underline{1\bar{2}}0\bar{1} = 0(2,\bar{1})(\bar{2},1)\bar{1}2\bar{1}0\bar{1} = (2,\bar{1})(\bar{2},1)0\bar{1}2\underline{\bar{1}0\bar{1}} = (2,\bar{1})(\bar{2},1)0\bar{1}2(0,\bar{1})(\bar{0},1)1\bar{0}$
$= (1,\bar{2},\bar{0})(\bar{1},2,0)\underline{\bar{1}0}21\bar{0} = (1,\bar{2},\bar{0})(\bar{1},2,0)(0,\bar{1})(\bar{0},1)1\bar{0}121\bar{0} = (2,\bar{1})(\bar{2},1)1\bar{0}\underline{121}\bar{0}$
$= (2,\bar{1})(\bar{2},1)1\bar{0}(2,1)(\bar{2},\bar{1})\bar{1}\bar{2}\bar{0} = (1,\bar{1})(\bar{2},2)2\bar{0}\bar{1}\bar{2}\bar{0} \in [012\bar{0}1]$. ☐

Lemma-13

Proof. $01\underline{\bar{2}0\bar{2}} = 01(2,\bar{0})(\bar{2},0)2\bar{0} = (2,\bar{0})(\bar{2},0)\bar{2}12\bar{0} \in [01\bar{0}2]$. ☐

Lemma-14

Proof. $0\underline{1\bar{2}}\bar{1}01 = 0(2,\bar{1})(\bar{2},1)\bar{1}2\bar{1}\bar{1}01 = (2,\bar{1})(\bar{2},1)0\bar{1}2\underline{101} = (2,\bar{1})(\bar{2},1)0\bar{1}2(1,0)(\bar{1},\bar{0})\bar{1}\bar{0}$
$= (1,\bar{2},0)(\bar{1},2,\bar{0})\underline{1\bar{0}}2\bar{1}\bar{0} = (1,\bar{2},0)(\bar{1},2,\bar{0})(0,\bar{1})(\bar{0},1)\bar{1}0\bar{1}2\bar{1}\bar{0} = (0,\bar{0})(1,\bar{2},\bar{1},2)\bar{1}0\underline{\bar{1}2\bar{1}}\bar{0}$
$= (0,\bar{0})(1,\bar{2},\bar{1},2)\bar{1}0(2,\bar{1})(\bar{2},1)1\bar{2}\bar{0} = (2,\bar{2})(\bar{0},0)201\underline{\bar{2}\bar{0}} = (2,\bar{2})(\bar{0},0)201(2,0)(\bar{2},\bar{0})202$
$= (2,\bar{0})(\bar{2},0)0\underline{212}02 = (2,\bar{0})(\bar{2},0)0(2,1)(\bar{2},\bar{1})\bar{2}\bar{1}02 = (1,2,\bar{0})(\bar{1},\bar{2},0)0\bar{2}\bar{1}02 \in [0120\bar{1}].$ □

Lemma-15

Proof. $0\underline{1\bar{2}}\bar{1}\bar{0}1 = 0(2,\bar{1})(\bar{2},1)\bar{1}2\bar{1}\bar{1}\bar{0}1 = (2,\bar{1})(\bar{2},1)0\bar{1}2\underline{1\bar{0}1} = (2,\bar{1})(\bar{2},1)0\bar{1}2(0,\bar{1})(\bar{0},1)\bar{1}0$
$= (1,\bar{2},\bar{0})(\bar{1},2,0)\bar{1}02\bar{1}0 \in [01\bar{2}01].$ □

Lemma-16

Proof. $01\bar{2}\underline{010} = 01\bar{2}(1,0)(\bar{1},\bar{0})\bar{0}\bar{1} = (1,0)(\bar{1},\bar{0})10\bar{2}\bar{0}\bar{1} \in [012\bar{1}\bar{0}].$ □

Lemma-17

Proof. $01\bar{2}\underline{0\bar{1}} = 01\bar{2}(0,\bar{1})(\bar{0},1)\bar{0}1\bar{0} = (0,\bar{1})(\bar{0},1)\bar{1}\underline{0\bar{2}\bar{0}}1\bar{0} = (0,\bar{1})(\bar{0},1)\bar{1}(2,0)(\bar{2},\bar{0})021\bar{0}$
$= (0,\bar{1},2)(\bar{0},1,\bar{2})\bar{1}0\underline{21}\bar{0} = (0,\bar{1},2)(\bar{0},1,\bar{2})\bar{1}0(2,1)(\bar{2},\bar{1})\bar{2}\bar{1}\bar{2}\bar{0} = (1,\bar{1})(0,\bar{2},\bar{0},2)\underline{\bar{2}0\bar{2}}\bar{1}\bar{2}\bar{0}$
$= (1,\bar{1})(0,\bar{2},\bar{0},2)(0,\bar{2})(\bar{0},2)2\bar{0}\bar{1}\bar{2}\bar{0} = (1,\bar{1})(\bar{2},2)2\bar{0}\bar{1}\bar{2}\bar{0} \in [012\bar{0}1].$ □

Lemma-18

Proof. $01\underline{\bar{2}0}12 = 01(0,\bar{2})(\bar{0},2)2\bar{0}212 = (0,\bar{2})(\bar{0},2)\bar{2}12\bar{0}\underline{212} = (0,\bar{2})(\bar{0},2)\bar{2}12\bar{0}(1,2)(\bar{1},\bar{2})\bar{2}\bar{1}$
$= (1,2,\bar{0})(\bar{1},\bar{2},0)\bar{1}21\underline{\bar{0}\bar{2}}\bar{1} = (1,2,\bar{0})(\bar{1},\bar{2},0)\bar{1}21(2,0)(\bar{2},\bar{0})020\bar{1}$
$= (2,\bar{2})(1,0,\bar{1},\bar{0})\bar{1}\underline{010}20\bar{1} = (2,\bar{2})(1,0,\bar{1},\bar{0})\bar{1}(1,0)(\bar{1},\bar{0})\bar{0}\bar{1}20\bar{1} = (0,\bar{0})(\bar{2},2)\bar{0}\bar{0}\bar{1}20\bar{1}$
$= (0,\bar{0})(\bar{2},2)0\bar{1}20\bar{1} \in [01\bar{2}01].$ □

Lemma-19

Proof. $01\bar{2}\underline{01}\bar{0} = 01\bar{2}(1,0)(\bar{1},\bar{0})\bar{0}\bar{1}\bar{0}\bar{0} = (1,0)(\bar{1},\bar{0})1\underline{0\bar{2}}\bar{0}\bar{1}0 = (1,0)(\bar{1},\bar{0})1(0,\bar{2})(\bar{0},2)\bar{0}2\bar{0}\bar{0}\bar{1}0$
$= (0,1,\bar{2})(\bar{0},\bar{1},2)1\bar{0}2\underline{0\bar{1}0} = (0,1,\bar{2})(\bar{0},\bar{1},2)1\bar{0}2(0,\bar{1})(\bar{0},1)\bar{0}1$
$= (0,\bar{0})(1,\bar{2},\bar{1},2)\bar{0}12\bar{0}1 \in [01\bar{2}01].$ □

Lemma-20

Proof. $012\bar{1}\bar{0}1 = 0\underline{121}1\bar{0}1 = 0(1,2)(\bar{1},\bar{2})\bar{1}\bar{2}1\bar{0}1 = (1,2)(\bar{1},\bar{2})0\bar{1}\bar{2}\underline{1\bar{0}1}$
$= (1,2)(\bar{1},\bar{2})0\bar{1}\bar{2}(0,\bar{1})(\bar{0},1)\bar{1}0 = (1,2,\bar{0})(\bar{1},\bar{2},0)\bar{1}0\bar{2}\bar{1}0 \in [01201].$ □

Lemma-21

Proof. $0\underline{12}\bar{1}01 = 0(1,2)(\bar{1},\bar{2})\bar{1}\bar{2}\bar{1}\bar{1}01 = (1,2)(\bar{1},\bar{2})0\bar{1}\bar{2}\underline{101} = (1,2)(\bar{1},\bar{2})0\bar{1}\bar{2}(1,0)(\bar{1},\bar{0})\bar{1}\bar{0}$
$= (1,2,0)(\bar{1},\bar{2},\bar{0})1\bar{0}\bar{2}\bar{1}\bar{0} \in [012\bar{0}1]$. □

Lemma-22

Proof. $012\underline{\bar{0}1}0 = 012(0,\bar{1})(\bar{0},1)0\bar{1}00 = (0,\bar{1})(\bar{0},1)\bar{1}\underline{\bar{0}2}0\bar{1}\bar{0} = (0,\bar{1})(\bar{0},1)\bar{1}(0,\bar{2})(\bar{0},2)0\bar{2}00\bar{1}\bar{0}$
$= (1,2,\bar{0})(\bar{1},\bar{2},0)\bar{1}0\bar{2}\underline{\bar{0}\bar{1}\bar{0}} = (1,2,\bar{0})(\bar{1},\bar{2},0)\bar{1}0\bar{2}(1,0)(\bar{1},\bar{0})01$
$= (0,\bar{0})(1,2,\bar{1},\bar{2})\bar{0}1\bar{2}01 \in [012\bar{0}1]$. □

Lemma-23

Proof. $\underline{01}2\bar{0}12 = (1,0)(\bar{1},\bar{0})\bar{0}\bar{1}\underline{\bar{0}2\bar{0}}12 = (1,0)(\bar{1},\bar{0})\bar{0}\bar{1}(0,\bar{2})(\bar{0},2)0\bar{2}12$
$= (0,1,\bar{2})(\bar{0},\bar{1},2)2\underline{\bar{1}0}\bar{2}12 = (0,1,\bar{2})(\bar{0},\bar{1},2)2(0,\bar{1})(\bar{0},1)1\bar{0}1\bar{2}12$
$= (0,\bar{0})(2,1,\bar{1},\bar{2})21\bar{0}\underline{1\bar{2}1}2 = (0,\bar{0})(2,1,\bar{1},\bar{2})21\bar{0}(2,\bar{1})(\bar{2},1)\bar{1}22$
$= (0,\bar{0})(\bar{2},2)\bar{1}\bar{2}\bar{0}\bar{1}\bar{2} \in [01\bar{2}01]$. □

Lemma-24

Proof. $012\underline{\bar{0}1\bar{0}} = 012(0,\bar{1})(\bar{0},1)0\bar{1} = (0,\bar{1})(\bar{0},1)\bar{1}\bar{0}20\bar{1} \in [012\bar{1}0]$. □

Lemma-25

Proof. $01\underline{2\bar{0}}1\bar{2} = 01(0,\bar{2})(\bar{0},2)\bar{2}0\bar{2}1\bar{2} = (0,\bar{2})(\bar{0},2)\underline{\bar{2}1\bar{2}}0\bar{2}1\bar{2} = (0,\bar{2})(\bar{0},2)(1,\bar{2})(\bar{1},2)2\bar{1}0\bar{2}1\bar{2}$
$= (1,\bar{2},0)(\bar{1},2,\bar{0})2\bar{1}0\underline{\bar{2}1\bar{2}} = (1,\bar{2},0)(\bar{1},2,\bar{0})2\bar{1}0(1,\bar{2})(\bar{1},2)2\bar{1} = (0,\bar{2})(\bar{0},2)\bar{1}\underline{202}\bar{1}$
$= (0,\bar{2})(\bar{0},2)\bar{1}(2,0)(\bar{2},\bar{0})\bar{2}\bar{0}\bar{1} = (0,\bar{0})(\bar{2},2)\bar{1}\bar{2}\bar{0}\bar{1} \in [01\bar{2}0]$. □

Lemma-26

Proof. $012\underline{0\bar{1}0} = 012(0,\bar{1})(\bar{0},1)\bar{0}1 = (0,\bar{1})(\bar{0},1)\bar{1}\underline{\bar{0}2\bar{0}}1 = (0,\bar{1})(\bar{0},1)\bar{1}(0,\bar{2})(\bar{0},2)0\bar{2}1$
$= (0,\bar{1},\bar{2})(\bar{0},1,2)\bar{1}0\bar{2}1 \in [012\bar{0}]$. □

Lemma-27

Proof. $0\underline{12}0\bar{1}2 = 0(1,2)(\bar{1},\bar{2})\bar{1}\bar{2}\bar{1}0\bar{1}2 = (1,2)(\bar{1},\bar{2})0\bar{1}\bar{2}\underline{\bar{1}0\bar{1}}2 = (1,2)(\bar{1},\bar{2})0\bar{1}\bar{2}(0,\bar{1})(\bar{0},1)1\bar{0}2$
$= (1,2,\bar{0})(\bar{1},\bar{2},0)\bar{1}0\underline{\bar{2}1}\bar{0}2 = (1,2,\bar{0})(\bar{1},\bar{2},0)\bar{1}0(2,\bar{1})(\bar{2},1)2\bar{1}2\bar{0}2$
$= (1,\bar{1})(2,\bar{0},\bar{2},0)\underline{202}\bar{1}2\bar{0}2 = (1,\bar{1})(2,\bar{0},\bar{2},0)(0,2)(\bar{0},\bar{2})\bar{2}\bar{0}\bar{1}2\bar{0}2 = (1,\bar{1})(\bar{2},2)\bar{2}\bar{0}\bar{1}\underline{2\bar{0}2}$
$= (1,\bar{1})(\bar{2},2)\bar{2}\bar{0}\bar{1}(0,\bar{2})(\bar{0},2)\bar{2}0 = (1,\bar{1})(2,0,\bar{2},\bar{0})02\bar{1}\bar{2}0 \in [01\bar{2}\bar{1}0]$. □

Lemma-28

Proof. $0\underline{12}0\bar{1}\bar{0} = 0(1,2)(\bar{1},\bar{2})\bar{1}\bar{2}\bar{1}0\bar{1}\bar{0} = (1,2)(\bar{1},\bar{2})0\bar{1}\bar{2}\underline{\bar{1}0\bar{1}}\bar{0} = (1,2)(\bar{1},\bar{2})0\bar{1}\bar{2}(1,\bar{0})(\bar{1},0)1\bar{0}\bar{0}$
$= (1,2,\bar{0})(\bar{1},\bar{2},0)\bar{1}0\bar{2}10 \in [012\bar{0}1]$. ☐

Lemma-29

Proof. $01\underline{20}\bar{1}\bar{2} = 01(0,2)(\bar{0},\bar{2})\bar{2}\bar{0}\bar{2}\bar{1}\bar{2} = (0,2)(\bar{0},\bar{2})21\bar{2}\bar{0}\underline{\bar{2}\bar{1}\bar{2}} = (0,2)(\bar{0},\bar{2})21\bar{2}\bar{0}(1,2)(\bar{1},\bar{2})21$
$= (1,2,0)(\bar{1},\bar{2},\bar{0})\underline{12}\bar{1}\bar{0}21 = (1,2,0)(\bar{1},\bar{2},\bar{0})(1,2)(\bar{1},\bar{2})\bar{1}\bar{2}\bar{1}\bar{1}\bar{0}21 = (0,2)(\bar{0},\bar{2})\bar{1}\underline{\bar{2}\bar{1}}\bar{0}21$
$= (0,2)(\bar{0},\bar{2})\bar{1}(1,\bar{2})(\bar{1},2)2\bar{1}2\bar{0}21 = (2,0,\bar{1})(\bar{2},\bar{0},1)22\bar{1}2\bar{0}21 = (2,0,\bar{1})(\bar{2},\bar{0},1)\bar{2}\bar{1}\underline{2\bar{0}2}1$
$= (2,0,\bar{1})(\bar{2},\bar{0},1)\bar{2}\bar{1}(0,\bar{2})(\bar{0},2)\bar{2}01 = (2,\bar{2})(1,0,\bar{1},\bar{0})0\bar{1}\bar{2}01 \in [0120\bar{1}]$. ☐

Lemma-30

Proof. $012\underline{010} = 012(0,1)(\bar{0},\bar{1})\bar{0}\bar{1} = (0,1)(\bar{0},\bar{1})102\bar{0}\bar{1} \in [012\bar{1}\bar{0}]$. ☐

Lemma-31

Proof. $01201\bar{0} = 012\underline{010}0 = 012(0,1)(\bar{0},\bar{1})\bar{0}\bar{1}0 = (0,1)(\bar{0},\bar{1})1\underline{02}\bar{0}\bar{1}0$
$= (0,1)(\bar{0},\bar{1})1(0,2)(\bar{0},\bar{2})\bar{0}\bar{2}\bar{0}\bar{0}\bar{1}0 = (1,2,0)(\bar{1},\bar{2},\bar{0})1\bar{0}\bar{2}\underline{0\bar{1}0}$
$= (1,2,0)(\bar{1},\bar{2},\bar{0})1\bar{0}\bar{2}(0,\bar{1})(\bar{0},1)\bar{0}1 = (0,\bar{0})(1,2,\bar{1},\bar{2})\bar{0}1\bar{2}\bar{0}1 \in [012\bar{1}\bar{0}]$. ☐

Lemma-32

Proof. $01\underline{20}1\bar{2} = 01(0,2)(\bar{0},\bar{2})\bar{2}\bar{0}\bar{2}1\bar{2} = (0,2)(\bar{0},\bar{2})21\bar{2}\bar{0}\underline{\bar{2}1\bar{2}} = (0,2)(\bar{0},\bar{2})21\bar{2}\bar{0}(2,\bar{1})(\bar{2},1)2\bar{1}$
$= (2,0,\bar{1})(\bar{2},\bar{0},1)\bar{1}\underline{\bar{2}1}\bar{0}2\bar{1} = (2,0,\bar{1})(\bar{2},\bar{0},1)\bar{1}(2,\bar{1})(\bar{2},1)2\bar{1}2\bar{0}2\bar{1} = (0,2)(\bar{0},\bar{2})22\bar{1}2\bar{0}2\bar{1}$
$= (0,2)(\bar{0},\bar{2})\bar{2}\bar{1}\underline{2\bar{0}2}\bar{1} = (0,2)(\bar{0},\bar{2})\bar{2}\bar{1}(2,\bar{0})(\bar{2},0)\bar{2}0\bar{1} = (0,0)(\bar{2},\bar{2})0\bar{1}\bar{2}0\bar{1} \in [01201]$. ☐

Lemma-33

Proof. $012\underline{01}2\bar{0} = 012(0,1)(\bar{0},\bar{1})\bar{0}\bar{1}\bar{0}2\bar{0} = (0,1)(\bar{0},\bar{1})102\bar{0}\bar{1}\underline{\bar{0}2\bar{0}}$
$= (0,1)(\bar{0},\bar{1})102\bar{0}\bar{1}(2,\bar{0})(\bar{2},0)0\bar{2} = (0,1,\bar{2})(\bar{0},\bar{1},2)1\bar{2}\underline{\bar{0}2}\bar{1}0\bar{2}$
$= (0,1,\bar{2})(\bar{0},\bar{1},2)1\bar{2}(2,\bar{0})(\bar{2},0)0\bar{2}0\bar{1}0\bar{2} = (0,1)(\bar{0},\bar{1})100\bar{2}0\bar{1}0\bar{2}$
$= (0,1)(\bar{0},\bar{1})1\bar{0}\bar{2}\underline{0\bar{1}0}\bar{2} = (0,1)(\bar{0},\bar{1})1\bar{0}\bar{2}(1,\bar{0})(\bar{1},0)\bar{0}1\bar{2} = (0,0)(\bar{1},\bar{1})\bar{0}1\bar{2}\bar{0}1\bar{2} \in [012012]$. ☐

Lemma-34

Proof. $01\bar{2}\underline{01}\bar{2}\bar{0} = 01\bar{2}(0,1)(\bar{0},\bar{1})\bar{0}\bar{1}\bar{0}\bar{2}\bar{0} = (0,1)(\bar{0},\bar{1})10\bar{2}\bar{0}\bar{1}\underline{\bar{0}\bar{2}\bar{0}}$

$= (0,1)(\bar{0},\bar{1})10\bar{2}\bar{0}\bar{1}(0,2)(\bar{0},\bar{2})02 = (1,2,0)(\bar{1},\bar{2},\bar{0})12\underline{\bar{0}\bar{2}}\bar{1}02$

$= (1,2,0)(\bar{1},\bar{2},\bar{0})12(0,2)(\bar{0},\bar{2})020\bar{1}02 = (0,1)(\bar{0},\bar{1})10020\bar{1}02$

$= (0,1)(\bar{0},\bar{1})1\bar{0}2\underline{0\bar{1}0}2 = (0,1)(\bar{0},\bar{1})1\bar{0}2(1,\bar{0})(\bar{1},0)\bar{0}12 = (0,\bar{0})(\bar{1},1)\bar{0}12\bar{0}12 \in [01\bar{2}01\bar{2}]$. $\square$

Chapter 7

Factoring the Center of $3^{\cdot}A_7$

7.1 Double coset enumeration of G over S_4

The center of $3^{\cdot}A_7$ constructed in the previous chapter is given by $< (1,0,2)(\bar{1},\bar{0},\bar{2})t_0t_1t_2 = \bar{t_0}\bar{t_2}\bar{t_1}\bar{t_0} >$. We now factor

$$G = \frac{3^{*3}:_m S_4}{[(1,\bar{2})(\bar{1},2)t_1]^5}$$

by the relator and construct A_7 as follows.

$$G \cong \frac{3^{*3}:_m S_4}{[(1,\bar{2})(\bar{1},2)t_1]^5, [(1,0,2)(\bar{1},\bar{0},\bar{2})t_0t_1t_2 = \bar{t_0}\bar{t_2}\bar{t_1}\bar{t_0}]}$$

The symmetric presentation of the progenitor $3^{*3} :_m S_4$ can be given by:

$$3^{*3} :_m S_4 \cong < x, y, t | x^4, y^2, (yx)^3, t^3, [ty], (x^2t)^2 >$$

The control group is $N = S_4 \cong < x, y | x^4 = y^2 = (yx)^3 >$. The action on the symmetric generators is given by $x \sim (0,1,\bar{0},\bar{1})(2,\bar{2})$, $y \sim (1,\bar{2})(2,\bar{1})$. The relations are $[(1,\bar{2})(\bar{1},2)t_1]^5$ and $(1,0,2)(\bar{1},\bar{0},\bar{2})t_0t_1t_2 = \bar{t_0}\bar{t_2}\bar{t_1}\bar{t_0}$.

First, we study our relations.

Relation-I

Let $\pi = (1,\bar{2})(2,\bar{1})$, then $(\pi t_1)^5 = 1$

$$
\begin{aligned}
(\pi t_1)^5 &= 1 \\
\pi t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1 &= 1 \\
\pi t_1 \pi t_1 \pi t_1 \pi \pi \pi^{-1} t_1 \pi t_1 &= 1 \\
\pi t_1 \pi t_1 \pi t_1 \pi^2 (t_1)^{\pi} t_1 &= 1 \\
\pi t_1 \pi t_1 \pi \pi^2 \pi^{-2} t_1 \pi^2 (t_1)^{\pi} t_1 &= 1 \\
\pi t_1 \pi \pi^3 \pi^{-3} t_1 \pi^3 (t_1)^{\pi^2} (t_1)^{\pi} t_1 &= 1 \\
\pi t_1 \pi^4 (t_1)^{\pi^3} (t_1)^{\pi^2} (t_1)^{\pi} t_1 &= 1 \\
\pi \pi^4 \pi^{-4} t_1 \pi^4 (t_1)^{\pi^3} (t_1)^{\pi^2} (t_1)^{\pi} t_1 &= 1 \\
\pi^5 (t_1)^{\pi^4} (t_1)^{\pi^3} (t_1)^{\pi^2} (t_1)^{\pi} t_1 &= 1
\end{aligned}
$$

Since, $\pi = (1,\bar{2})(2,\bar{1})$
$\Rightarrow \pi^2 = (1,\bar{2})(2,\bar{1})(1,\bar{2})(2,\bar{1}) = 1.$
$\Rightarrow \pi^3 = (1,\bar{2})(2,\bar{1}).$
$\Rightarrow \pi^4 = 1.$
$\Rightarrow \pi^5 = (1,\bar{2})(2,\bar{1}).$

$$
\begin{aligned}
\pi^5 (t_1)^{\pi^4} (t_1)^{\pi^3} (t_1)^{\pi^2} (t_1)^{\pi} t_1 &= 1 \\
\Leftrightarrow \pi^5 t_1 \bar{t_2} t_1 \bar{t_2} t_1 &= 1 \\
\Leftrightarrow \pi^5 t_1 \bar{t_2} t_1 &= \bar{t_1} t_2
\end{aligned}
$$

Relation-II

$(1,0,2)(\bar{1},\bar{0},\bar{2})t_0 t_1 t_2 = \bar{t_0}\bar{t_2}\bar{t_1}\bar{t_0}.$

Double coset enumeration of G over S_4.

Unless otherwise specified, all the lemmas using in Chapter-7 can be found from pages 107-108.

We note that,

$$NeN = \{Nen|n \in N\} = \{Nn|n \in N\} = \{N\}.$$

Let [*] represents the double coset NeN. [*] has only one distinct right coset, namely N. Since, $N \cong S_4$ acts transitively on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$. Therefore, N has one orbit namely $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$. Now, we take one representative 0 from the orbit, and find which double coset Nt_0 belongs.

Clearly, $Nt_0 \in Nt_0N = \{Nt_0^n|n \in N\} = \{Nt_0, Nt_1, Nt_2, N\bar{t_0}, N\bar{t_1}, N\bar{t_2}\}$ and let us denote Nt_0N by [0]. Now, consider the coset stabilizer $N^{(0)}$. The coset stabilizer $N^{(0)}$ is equal to the point stabilizer N^0 and is given by:

$$N^{(0)} = N^0 = \{e, (1, \bar{2})(2, \bar{1}), (1, 2)(\bar{1}, \bar{2}), (1, \bar{1})(2, \bar{2})\}.$$

Therefore, the number of single cosets in [0] are at most

$$\frac{|N|}{|N^{(0)}|} = \frac{4!}{4} = \frac{24}{4} = 6.$$

The orbits of $N^{(0)}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}$, $\{\bar{0}\}$ and $\{1, 2, \bar{1}, \bar{2}\}$. We now take one representative from each orbit, namely 0, $\bar{0}$ and 1, and find the double coset to which Nt_0t_0, $Nt_0\bar{t_0}$ and Nt_0t_1 belong? However,

$Nt_0t_0 = N\bar{t_0} \in [0]$
$Nt_0\bar{t_0} = Ne \in [*]$
$Nt_0t_1 = Nt_0t_1 \in Nt_0t_1N$ (New double coset denote by [01])

Therefore, one symmetric generator goes back to the identity double coset [*]; one stays in [0]; four extend to new double coset [01].

Next, consider the new double coset [01].

$N^{(01)} = N^{01} =< e >$.
Therefore, the number of single cosets in the double coset [01] are at most

$\frac{|N|}{|N^{(01)}|} = \frac{4!}{1} = \frac{24}{1} = 24$. Each single coset has only one name.

By conjugating Nt_0t_1 with the elements of transversal of $N^{(01)}$, we obtain the 24 single cosets of double coset [01]:
$01, \bar{2}1, 21, \bar{0}1, 12, \bar{1}2, \bar{0}2, 02, \bar{1}0, 10, 20, \bar{2}0, \bar{0}\bar{1}, 0\bar{1}, \bar{2}\bar{1}, 2\bar{1}, 0\bar{2}, \bar{0}\bar{2}, 1\bar{2}, \bar{1}\bar{2}, 1\bar{0}, \bar{2}\bar{0}, \bar{1}\bar{0}, 2\bar{0}$.

The orbits of $N^{(01)}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1}\}$ and $\{\bar{2}\}$. Now, take one representative from each orbit, namely 0, 1, 2, $\bar{0}$, $\bar{1}$ and $\bar{2}$, then find the double cosets to which $Nt_0t_1t_0$, $Nt_0t_1t_1$, $Nt_0t_1t_2$, $Nt_0t_1\bar{t_0}$, $Nt_0t_1\bar{t_1}$ and $Nt_0t_1\bar{t_2}$ belong? However,

$Nt_0t_1t_0 = Nt_0t_1t_0 \in [01]$ **(By Lemma-1).**
$Nt_0t_1t_1 = Nt_0\bar{t_1} \in [0\bar{1}]$.
$Nt_0t_1t_2 = Nt_0t_1t_2 \in Nt_0t_1t_2N$ (New double coset denote by [012]).
$Nt_0t_1\bar{t_0} = Nt_0t_1\bar{t_0} \in Nt_0t_1\bar{t_0}N$ (New double coset denote by $[01\bar{0}]$).
$Nt_0t_1\bar{t_1} = Nt_0 \in [0]$.
$Nt_0t_1\bar{t_2} = Nt_0t_1\bar{t_2} \in Nt_0t_1\bar{t_2}N$ (New double coset denote by $[01\bar{2}]$).

Consider the new double coset [012].

$N^{(012)} = N^{012} =< e >$.
Therefore, the number of single cosets in the double coset [012] are at most

$\frac{|N|}{|N^{(012)}|} = \frac{4!}{1} = \frac{24}{1} = 24$. Each single coset has only one name.

By conjugating $Nt_0t_1t_2$ with the elements of transversal of $N^{(012)}$, we obtain the 24 single cosets of double coset [012]:
$012, \bar{2}1\bar{0}, 210, \bar{0}1\bar{2}, 120, \bar{1}2\bar{0}, \bar{0}2\bar{1}, 021, \bar{1}0\bar{2}, 102, 201, \bar{2}0\bar{1}, \bar{0}\bar{1}2, 0\bar{1}\bar{2}, \bar{2}\bar{1}0, 2\bar{1}\bar{0}, 0\bar{2}\bar{1}, \bar{0}\bar{2}1, 1\bar{2}\bar{0}$,
$\bar{1}\bar{2}0, 1\bar{0}\bar{2}, \bar{2}\bar{0}1, \bar{1}\bar{0}2, 2\bar{0}\bar{1}$.

The orbits of $N^{(012)}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1}\}$ and $\{\bar{2}\}$. Now, take one representative from each orbit, namely 0, 1, 2, $\bar{0}$, $\bar{1}$ and $\bar{2}$, then find the double cosets to which $Nt_0t_1t_2t_0$, $Nt_0t_1t_2t_1$, $Nt_0t_1t_2t_2$, $Nt_0t_1t_2\bar{t_0}$, $Nt_0t_1t_2\bar{t_1}$ and $Nt_0t_1t_2\bar{t_2}$ belong? However,

$Nt_0t_1t_2t_0 = Nt_0t_1t_2t_0 \in [01\bar{2}$ **(By Lemma-2).**
$Nt_0t_1t_2t_1 = Nt_0t_1t_2t_1 \in [012]$ **(By Lemma-3).**

$Nt_0t_1t_2t_2 = Nt_0t_1\bar{t_2} \in [01\bar{2}]$.
$Nt_0t_1t_2\bar{t_0} = Nt_0t_1t_2\bar{t_0} \in Nt_0t_1t_2\bar{t_0}N$ (New double coset denote by $[012\bar{0}]$).
$Nt_0t_1t_2\bar{t_1} = Nt_0t_1t_2\bar{t_1} \in Nt_0t_1t_2\bar{t_1}N$ (New double coset denote by $[012\bar{1}]$).
$Nt_0t_1t_2\bar{t_2} = Nt_0t_1 \in [01]$.

Consider the double coset $[01\bar{0}]$
Claim: $01\bar{0} \sim 10\bar{1} \sim \bar{0}\bar{1}0$

Proof. By using our relation conjugates with elements of N, we have
1. $0\underline{1\bar{0}} = 0(\bar{1},0)(1,\bar{0})\bar{1}0\bar{1} = (\bar{1},0)(1,\bar{0})\bar{1}\bar{1}0\bar{1} = (\bar{1},0)(1,\bar{0})10\bar{1}$.

2. $\underline{01}\bar{0} = (0,1)(\bar{0},\bar{1})\bar{0}\bar{1}\bar{0}\bar{0} = (0,1)(\bar{0},\bar{1})\bar{0}\bar{1}0$. □

Since $N^{(01\bar{0})} \geq N^{01\bar{0}} =< e >$ and
$N(t_0t_1\bar{t_0})^{\{(1,0)(\bar{1},\bar{0})\}} = Nt_1t_0\bar{t_1} \Rightarrow (1,0)(\bar{1},\bar{0}) \in N^{(01\bar{0})}$
$N(t_0t_1\bar{t_0})^{\{(1,\bar{1})(0,\bar{0})\}} = N\bar{t_0}\bar{t_1}t_0 \Rightarrow (1,\bar{1})(0,\bar{0}) \in N^{(01\bar{0})}$
$\Rightarrow N^{(01\bar{0})} =< (1,0)(\bar{1},\bar{0}),(1,\bar{1})(0,\bar{0}) >$
$= \{e,(1,0)(\bar{1},\bar{0}),(1,\bar{1})(0,\bar{0}),(1,\bar{0})(0,\bar{1})\}$.
Therefore, the number of single cosets in the double coset $[01\bar{0}]$ are at most

$\frac{|N|}{|N^{(01\bar{0})}|} = \frac{4!}{4} = \frac{24}{4} = 6$. Each single coset has four different names.

Now, take $(01\bar{0})$ conjugates with the elements of $N^{(01\bar{0})}$, we obtain the following relations:

$$01\bar{0} \sim 10\bar{1} \sim \bar{0}\bar{1}0 \sim \bar{1}\bar{0}1$$

By conjugating $(01\bar{0} \sim 10\bar{1} \sim \bar{0}\bar{1}0 \sim \bar{1}\bar{0}1)$ with the elements of transversal of $N^{(01\bar{0})}$, we obtain the following six single cosets of double coset $[01\bar{0}]$:

$$01\bar{0} \sim 10\bar{1} \sim \bar{0}\bar{1}0 \sim \bar{1}\bar{0}1$$
$$\bar{0}10 \sim 1\bar{0}\bar{1} \sim 0\bar{1}\bar{0} \sim \bar{1}01$$
$$0\bar{2}\bar{0} \sim \bar{2}02 \sim \bar{0}20 \sim 2\bar{0}\bar{2}$$
$$\bar{0}\bar{2}0 \sim \bar{2}\bar{0}2 \sim 02\bar{0} \sim 20\bar{2}$$
$$12\bar{1} \sim 21\bar{2} \sim \bar{1}\bar{2}1 \sim \bar{2}\bar{1}2$$
$$\bar{1}21 \sim 2\bar{1}\bar{2} \sim 1\bar{2}\bar{1} \sim \bar{2}12$$

The orbits of $N^{(01\bar{0})}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{2\}$, $\{\bar{2}\}$ and $\{1,0,\bar{1},\bar{0}\}$. Now, take one representative from each orbit, namely 2, $\bar{2}$ and 0, then find the double coset to which $Nt_0t_1\bar{t_0}t_2$, $Nt_0t_1\bar{t_0}\bar{t_2}$ and $Nt_0t_1\bar{t_0}t_0$ belong? However,

$Nt_0t_1\bar{t_0}t_2 = Nt_0t_1\bar{t_0}t_2 \in [012\bar{1}]$ **(By Lemma-4).**
$Nt_0t_1\bar{t_0}\bar{t_2} = Nt_0t_1\bar{t_0}\bar{t_2} \in Nt_0t_1\bar{t_0}\bar{t_2}N$ (New double coset denote by $[01\bar{0}\bar{2}]$).
$Nt_0t_1\bar{t_0}t_0 = Nt_0t_1 \in [01]$.

Consider the double coset $[01\bar{2}]$.

$N^{(01\bar{2})} = N^{01\bar{2}} =< e >$.
Therefore, the number of single cosets in the double coset $[01\bar{2}]$ are at most

$\frac{|N|}{|N^{(01\bar{2})}|}=\frac{4!}{1} = \frac{24}{1} = 24$. Each single coset has only one name.

By conjugating $(01\bar{2})$ with the elements of transversal of $N^{(01\bar{2})}$, we obtain the following 24 single cosets of double coset $[01\bar{2}]$:
$01\bar{2}, \bar{2}10, 21\bar{0}, \bar{0}12, 12\bar{0}, \bar{1}20, \bar{0}21, 02\bar{1}, \bar{1}02, 10\bar{2}, 20\bar{1}, \bar{2}01, \bar{0}\bar{1}\bar{2}, 0\bar{1}2, \bar{2}\bar{1}\bar{0}, 2\bar{1}0, 0\bar{2}1, \bar{0}\bar{2}\bar{1}, 1\bar{2}0,$
$\bar{1}\bar{2}\bar{0}, 1\bar{0}2, \bar{2}\bar{0}\bar{1}, \bar{1}\bar{0}\bar{2}, 2\bar{0}1$.

The orbits of $N^{(01\bar{2})}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1}\}$ and $\{\bar{2}\}$. Now, take one representative from each orbit, namely 0, 1, 2, $\bar{0}$, $\bar{1}$ and $\bar{2}$, then find the double cosets to which $Nt_0t_1\bar{t_2}t_0$, $Nt_0t_1\bar{t_2}t_1$, $Nt_0t_1\bar{t_2}t_2$, $Nt_0t_1\bar{t_2}\bar{t_0}$, $Nt_0t_1\bar{t_2}\bar{t_1}$ and $Nt_0t_1\bar{t_2}\bar{t_2}$ belong? However,

$Nt_0t_1\bar{t_2}t_0 = Nt_0t_1\bar{t_2}t_0 \in [012]$ **(By Lemma-5).**
$Nt_0t_1\bar{t_2}t_1 = Nt_0t_1\bar{t_2}t_1 \in [01\bar{2}]$ **(By Lemma-6).**
$Nt_0t_1\bar{t_2}t_2 = Nt_0t_1 \in [01]$.
$Nt_0t_1\bar{t_2}\bar{t_0} = Nt_0t_1\bar{t_2}\bar{t_0} \in [012\bar{0}]$ **(By Lemma-7).**
$Nt_0t_1\bar{t_2}\bar{t_1} = Nt_0t_1\bar{t_2}\bar{t_1} \in [01\bar{0}\bar{2}]$ **(By Lemma-8).**
$Nt_0t_1\bar{t_2}\bar{t_2} = Nt_0t_1t_2 \in [012]$.

Consider the double coset $[012\bar{0}]$
Claim: $012\bar{0} \sim \bar{1}2\bar{0}1 \sim \bar{2}\bar{0}12$

Proof. By using our relation conjugates with elements of N, we have

1. $01 2\bar{0} = 0\underline{12}\underline{00} = 0(201)(\bar{2},\bar{0},\bar{1})\bar{1}0\bar{2}\bar{1}0 = (201)(\bar{2},\bar{0},\bar{1})1\bar{1}0\bar{2}\bar{1}0 = (201)(\bar{2},\bar{0},\bar{1})\bar{0}\bar{2}\underline{\bar{1}0}$
$= (201)(\bar{2},\bar{0},\bar{1})\bar{0}\bar{2}(1,\bar{0})(\bar{1},0)1\bar{0}1 = (2,\bar{1},\bar{2},1)(0,\bar{0})1\underline{\bar{2}1}\bar{0}1$
$= (2,\bar{1},\bar{2},1)(0,\bar{0})1(1,\bar{2})(\bar{1},2)2\bar{1}2\bar{0}1 = (1,\bar{1})(\bar{0},0)\bar{2}2\bar{1}2\bar{0}1 = (1,\bar{1})(\bar{0},0)\bar{1}2\bar{0}1.$

2. $01\underline{2\bar{0}} = 01(2,\bar{0})(\bar{2},0)\bar{2}0\bar{2} = (2,\bar{0})(\bar{2},0)\underline{\bar{2}1\bar{2}}0\bar{2} = (2,\bar{0})(\bar{2},0)(1,\bar{2})(\bar{1},2)2\bar{1}0\bar{2}$
$= (2,\bar{0},\bar{1})(\bar{2},0,1)2\bar{1}0\bar{2} = (2,\bar{0},\bar{1})(\bar{2},0,1)\underline{2\bar{1}0\bar{2}}2 = (2,\bar{0},\bar{1})(\bar{2},0,1)(1,\bar{0},\bar{2})(\bar{1},0,2)\bar{2}\bar{0}12$
$= (2,\bar{2})(\bar{0},0)\bar{2}\bar{0}12.$ □

Since $N^{(012\bar{0})} \geq N^{012\bar{0}} =< e >$ and
$N(t_0t_1t_2\bar{t_0})^{\{(1,2,\bar{0})(0,\bar{1},\bar{2})\}} = N\bar{t_1}t_2\bar{t_0}t_1 \Rightarrow (1,2,\bar{0})(0,\bar{1},\bar{2}) \in N^{(012\bar{0})}$
$N(t_0t_1t_2\bar{t_0})^{\{(1,\bar{0},2)(0,\bar{2},\bar{1})\}} = N\bar{t_2}\bar{t_0}t_1t_2 \Rightarrow (1,\bar{0},2)(0,\bar{2},\bar{1}) \in N^{(012\bar{0})}$
$\Rightarrow N^{(012\bar{0})} =< (1,2,\bar{0})(0,\bar{1},\bar{2}),(1,\bar{0},2)(0,\bar{2},\bar{1}) >$
$= \{e,(1,2,\bar{0})(0,\bar{1},\bar{2}),(1,\bar{0},2)(0,\bar{2},\bar{1})\}.$
Therefore, the number of single cosets in the double coset $[012\bar{0}]$ are at most

$\frac{|N|}{|N^{(0120)}|} = \frac{4!}{3} = \frac{24}{3} = 8$. Each single coset has three different names.

Now, take $(012\bar{0})$ conjugates with the elements of $N^{(012\bar{0})}$, we obtain the following relations:

$$012\bar{0} \sim \bar{1}2\bar{0}1 \sim \bar{2}\bar{0}12$$

By conjugating $(012\bar{0} \sim \bar{1}2\bar{0}1 \sim \bar{2}\bar{0}12)$ with the elements of transversal of $N^{(012\bar{0})}$, we obtain the following eight single cosets of double coset $[012\bar{0}]$:

$$\begin{gathered}
012\bar{0} \sim \bar{1}2\bar{0}1 \sim \bar{2}\bar{0}12 \\
1\bar{0}\bar{2}\bar{1} \sim 0\bar{2}\bar{1}\bar{0} \sim 2\bar{1}\bar{0}\bar{2} \\
\bar{0}\bar{1}20 \sim 120\bar{1} \sim \bar{2}0\bar{1}2 \\
\bar{1}0\bar{2}1 \sim \bar{0}\bar{2}10 \sim 210\bar{2} \\
\bar{0}2\bar{1}0 \sim \bar{2}\bar{1}02 \sim 102\bar{1} \\
201\bar{2} \sim \bar{0}1\bar{2}0 \sim \bar{1}\bar{2}01 \\
\bar{2}1\bar{0}2 \sim \bar{1}\bar{0}21 \sim 021\bar{0} \\
1\bar{2}\bar{0}\bar{1} \sim 2\bar{0}\bar{1}\bar{2} \sim 0\bar{1}\bar{2}\bar{0}
\end{gathered}$$

The orbits of $N^{(012\bar{0})}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{1,2,\bar{0}\}$ and $\{0,\bar{1},\bar{2}\}$. Now, take one representative from each orbit, namely $\bar{0}$ and 0, then find the double coset to which $Nt_0t_1t_2\bar{t_0}\bar{t_0}$ and $Nt_0t_1t_2\bar{t_0}t_0$ belong? However,

$Nt_0t_1t_2\bar{t_0}\bar{t_0} = Nt_0t_1t_2t_0 \in [01\bar{2}]$ **(By Lemma-2)**.
$Nt_0t_1t_2\bar{t_0}t_0 = Nt_0t_1t_2 \in [012]$.

Consider the double coset $[012\bar{1}]$
Claim: $012\bar{1} \sim 0\bar{1}\bar{2}1 \sim 021\bar{2}$

Proof. By using our relation conjugates with elements of N, we have
1. $012\bar{1} = 01\underline{211} = 01(1,2)(\bar{1},\bar{2})\bar{2}\bar{1}\bar{2}1 = (1,2)(\bar{1},\bar{2})02\bar{2}\bar{1}\bar{2}1 = (1,2)(\bar{1},\bar{2})0\bar{1}\bar{2}1$.

2. $01\underline{2\bar{1}} = 01(2,\bar{1})(\bar{2},1)\bar{2}1\bar{2} = (2,\bar{1})(\bar{2},1)0\bar{2}\bar{2}1\bar{2} = (2,\bar{1})(\bar{2},1)021\bar{2}$. □

Since $N^{(012\bar{1})} \geq N^{012\bar{1}} =< e >$ and
$N(t_0t_1t_2\bar{t_1})^{\{(1,\bar{1})(2,\bar{2})\}} = Nt_0\bar{t_1}\bar{t_2}t_1 \Rightarrow (1,\bar{1})(2,\bar{2}) \in N^{(012\bar{1})}$
$N(t_0t_1t_2\bar{t_1})^{\{(1,2)(\bar{1},\bar{2})\}} = Nt_0t_2t_1\bar{t_2} \Rightarrow (1,2)(\bar{1},\bar{2}) \in N^{(012\bar{1})}$
$\Rightarrow N^{(012\bar{1})} =< (1,\bar{1})(2,\bar{2}), (1,2)(\bar{1},\bar{2}) >$
$= \{e, (1,\bar{1})(2,\bar{2}), (1,2)(\bar{1},\bar{2}), (1,\bar{2})(2,\bar{1})\}$.
Therefore, the number of single cosets in the double coset $[012\bar{1}]$ are at most

$\frac{|N|}{|N^{(012\bar{1})}|} = \frac{4!}{4} = \frac{24}{4} = 6$. Each single coset has four different names.

Now, take $(012\bar{1})$ conjugates with the elements of $N^{(012\bar{1})}$, we obtain the following relations:

$$012\bar{1} \sim 0\bar{1}\bar{2}1 \sim 021\bar{2} \sim 0\bar{2}\bar{1}2$$

By conjugating $(012\bar{1} \sim 0\bar{1}\bar{2}1 \sim 021\bar{2} \sim 0\bar{2}\bar{1}2)$ with the elements of transversal of $N^{(012\bar{1})}$, we obtain the following six single cosets of double coset $[012\bar{1}]$:

$$012\bar{1} \sim 0\bar{1}\bar{2}1 \sim 021\bar{2} \sim 0\bar{2}\bar{1}2$$
$$1\bar{0}\bar{2}0 \sim 102\bar{0} \sim 1\bar{2}\bar{0}2 \sim 120\bar{2}$$
$$\bar{0}121 \sim \bar{0}1\bar{2}\bar{1} \sim \bar{0}2\bar{1}\bar{2} \sim \bar{0}\bar{2}12$$
$$\bar{2}010 \sim \bar{2}0\bar{1}\bar{0} \sim \bar{2}1\bar{0}\bar{1} \sim \bar{2}\bar{1}01$$

$$\bar{1}0\bar{2}\bar{0} \sim \bar{1}\bar{0}20 \sim \bar{1}\bar{2}02 \sim \bar{1}2\bar{0}\bar{2}$$
$$2\bar{0}\bar{1}0 \sim 201\bar{0} \sim 2\bar{1}\bar{0}1 \sim 210\bar{1}$$

The orbits of $N^{(012\bar{1})}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{\bar{0}\}$ and $\{1,2,\bar{1},\bar{2}\}$. Now, take one representative from each orbit, namely 0, $\bar{0}$ and 1, then find the double coset to which $Nt_0t_1t_2\bar{t_1}t_0$, $Nt_0t_1t_2\bar{t_1}\bar{t_0}$ and $Nt_0t_1t_2\bar{t_1}t_1$ belong? However,

$Nt_0t_1t_2\bar{t_1}t_0 = Nt_0t_1t_2\bar{t_1}t_0 \in [01\bar{0}\bar{2}]$ **(By Lemma-9).**
$Nt_0t_1t_2\bar{t_1}\bar{t_0} = Nt_0t_1t_2\bar{t_1}\bar{t_0} \in [01\bar{0}]$ **(By Lemma-10).**
$Nt_0t_1t_2\bar{t_1}t_1 = Nt_0t_1t_2 \in [012]$.

Consider the double coset $[01\bar{0}\bar{2}]$
Claim: $01\bar{0}\bar{2} \sim 10\bar{1}\bar{2} \sim \bar{0}\bar{1}0\bar{2}$

Proof. By using our relation conjugates with elements of N, we have
1. $0\underline{1\bar{0}}\bar{2} = 0(\bar{1},0)(1,\bar{0})\bar{1}0\bar{1}\bar{2} = (\bar{1},0)(1,\bar{0})\bar{1}\bar{1}0\bar{1}\bar{2} = (\bar{1},0)(1,\bar{0})10\bar{1}\bar{2}$.
2. $0\underline{1\bar{0}}\bar{2} = 0(\bar{1},0)(1,\bar{0})\bar{1}0\bar{1}\bar{2} = (\bar{1},0)(1,\bar{0})\bar{1}\bar{1}0\bar{1}\bar{2} = (\bar{1},0)(1,\bar{0})\underline{10}\bar{1}\bar{2}$
$= (\bar{1},0)(1,\bar{0})(1,0)(\bar{1},\bar{0})\bar{1}\bar{0}\bar{1}\bar{1}\bar{2} = (1,\bar{1})(0,\bar{0})\bar{1}\bar{0}1\bar{2}$. ◻

Since $N^{(01\bar{0}\bar{2})} \geq N^{01\bar{0}\bar{2}} =< e >$ and
$N(t_0t_1\bar{t_0}\bar{t_2})^{\{(1,0)(\bar{1},\bar{0})\}} = Nt_1t_0\bar{t_1}\bar{t_2} \Rightarrow (1,0)(\bar{1},\bar{0}) \in N^{(01\bar{0}\bar{2})}$
$N(t_0t_1\bar{t_0}\bar{t_2})^{\{(1,\bar{0})(0,\bar{1})\}} = N\bar{t_1}\bar{t_0}t_1\bar{t_2} \Rightarrow (1,\bar{0})(0,\bar{1}) \in N^{(01\bar{0}\bar{2})}$
$\Rightarrow N^{(01\bar{0}\bar{2})} =< (1,0)(\bar{1},\bar{0}),(1,\bar{0})(0,\bar{1}) >$
$= \{e,(1,0)(\bar{1},\bar{0}),(1,\bar{1})(0,\bar{0}),(1,\bar{0})(0,\bar{1})\}$.
Therefore, the number of single cosets in the double coset $[01\bar{0}\bar{2}]$ are at most

$\frac{|N|}{|N^{(01\bar{0}\bar{2})}|} = \frac{4!}{4} = \frac{24}{4} = 6$. Each single coset has four different names.

Now, take $(01\bar{0}\bar{2})$ conjugates with the elements of $N^{(01\bar{0}\bar{2})}$, we obtain the following relations:

$$01\bar{0}\bar{2} \sim 10\bar{1}\bar{2} \sim \bar{0}\bar{1}0\bar{2} \sim \bar{1}\bar{0}1\bar{2}$$

By conjugating $(01\bar{0}\bar{2} \sim 10\bar{1}\bar{2} \sim \bar{0}\bar{1}0\bar{2} \sim \bar{1}\bar{0}1\bar{2})$ with the elements of transversal of $N^{(01\bar{0}\bar{2})}$, we obtain the following six single cosets of double coset $[01\bar{0}\bar{2}]$:

$$0 1 \bar{0} \bar{2} \sim 1 0 \bar{1} \bar{2} \sim \bar{0} \bar{1} 0 \bar{2} \sim \bar{1} \bar{0} 1 \bar{2}$$
$$0 \bar{1} \bar{0} 2 \sim \bar{1} 0 1 2 \sim \bar{0} 1 0 2 \sim 1 \bar{0} \bar{1} 2$$
$$\bar{0} 2 0 1 \sim 2 \bar{0} \bar{2} 1 \sim 0 \bar{2} \bar{0} 1 \sim \bar{2} 0 2 1$$
$$0 2 \bar{0} \bar{1} \sim 2 0 \bar{2} \bar{1} \sim \bar{0} \bar{2} 0 \bar{1} \sim \bar{2} \bar{0} 2 \bar{1}$$
$$\bar{1} \bar{2} 1 \bar{0} \sim \bar{2} \bar{1} 2 \bar{0} \sim 1 2 \bar{1} \bar{0} \sim 2 1 \bar{2} \bar{0}$$
$$1 \bar{2} \bar{1} 0 \sim \bar{2} 1 2 0 \sim \bar{1} 2 1 0 \sim 2 \bar{1} \bar{2} 0$$

The orbits of $N^{(01\bar{0}\bar{2})}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{2\}$, $\{\bar{2}\}$ and $\{1, 0, \bar{1}, \bar{0}\}$. Now, take one representative from each orbit, namely 2, $\bar{2}$ and 1, then find the double coset to which $Nt_0t_1\bar{t_0}\bar{t_2}t_2$, $Nt_0t_1\bar{t_0}\bar{t_2}\bar{t_2}$ and $Nt_0t_1\bar{t_0}\bar{t_2}t_1$ belong? However,

$Nt_0t_1\bar{t_0}\bar{t_2}\bar{t_2} = Nt_0t_1\bar{t_0}t_2 \in [012\bar{1}]$ **(By Lemma-4).**
$Nt_0t_1\bar{t_0}\bar{t_2}t_1 = Nt_0t_1\bar{t_0}\bar{t_2}t_1 \in [01\bar{2}]$ **(By Lemma-11).**
$Nt_0t_1\bar{t_0}\bar{t_2}t_2 = Nt_0t_1\bar{t_0} \in [01\bar{0}]$.

Since all symmetric generators go back to the previous double cosets and none of them extend. Thus, we must have completed the double coset enumeration. All the information is summarized in the Cayley diagram below.

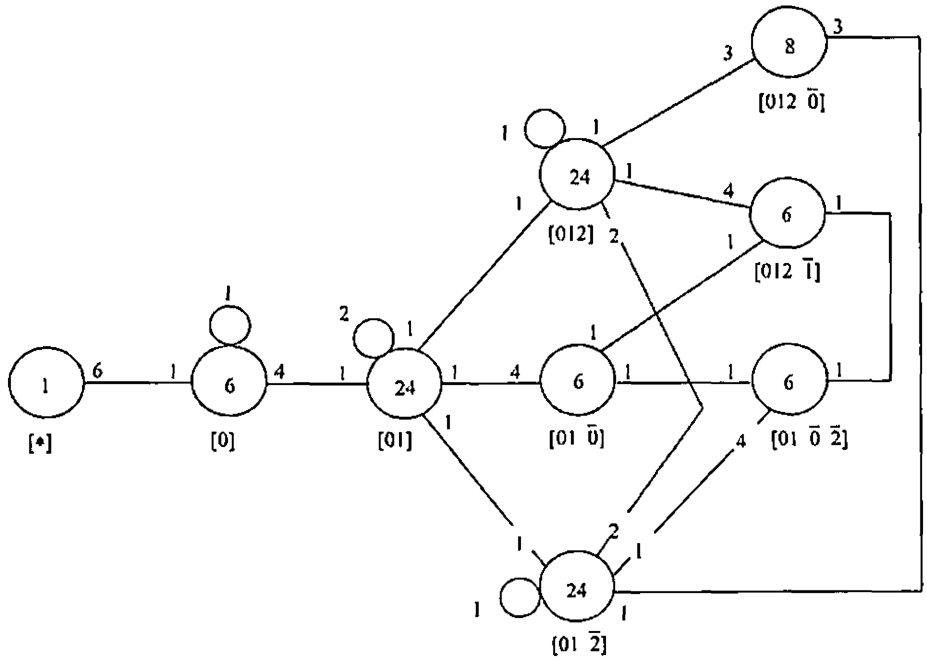

Figure 7.1: The Cayley diagram of A_7 over S_4.

The double coset enumeration shows that the index of $N \cong S_4$ is at most:
$|G| \leq (105)|N| = (105) \times 24 = 2,520$.

The homomorphic image of G.

Define $\widehat{\alpha} : G \to S_{105}$, where (105) is a number of single cosets from the Cayley diagram above gives the action of G on the 105 cosets. From the labeling table (too large to show), we compute the actions of x, y and t_0 on the 105 cosets.
$\widehat{\alpha}(x) = (2,4,3,5)(6,9)(7,14,11,15)...(88,95,90,94)(100,103,101,102)$.
$\widehat{\alpha}(y) = (4,6)(5,9)(7,12)(8,13)(10,18)(11,19)...(90,100)(94,103)(101,102)$.
$\widehat{\alpha}(t_0) = (1,2,3)(4,7,8)(5,10,11)(6,12,13)...(80,104,82)(83,105,85)$.
We now verify that $\widehat{\alpha}$ is a homomorphism.

1. We want to show that $\widehat{\alpha}(t_0)$ has exactly 6 conjugates under conjugation by $< \widehat{\alpha}(x), \widehat{\alpha}(y) >$.
   $\widehat{\alpha}(t_0)^{\widehat{\alpha}(x)} = (1,4,5)(2,17,15)(3,14,16)...(78,97,86)(79,99,87) = \widehat{\alpha}(t_1)$.
   $\widehat{\alpha}(t_1)^{\widehat{\alpha}(x)} = (1,3,2)(4,8,7)(5,11,10)...(80,82,104)(83,85,105) = \widehat{\alpha}(\bar{t_0})$.
   $\widehat{\alpha}(\bar{t_0})^{\widehat{\alpha}(x)} = (1,5,4)(2,15,17)(3,16,14)...(78,86,97)(79,87,99) = \widehat{\alpha}(\bar{t_1})$.
   $\widehat{\alpha}(\bar{t_1})^{\widehat{\alpha}(x)} = (1,2,3)(4,7,8)(5,10,11)...(80,104,82)(83,105,85) = \widehat{\alpha}(t_0)$.
   $\widehat{\alpha}(t_2)^{\widehat{\alpha}(x)} = (1,6,9)(2,27,24)(3,22,25)...(64,76,72)(65,79,73) = \widehat{\alpha}(\bar{t_2})$.
   $\widehat{\alpha}(\bar{t_2})^{\widehat{\alpha}(x)} = (1,9,6)(2,24,27)(3,25,22)...(64,72,76)(65,73,79) = \widehat{\alpha}(t_2)$.

   $\widehat{\alpha}(t_1)^{\widehat{\alpha}(y)} = (1,6,9)(2,27,24)(3,22,25)...(64,76,72)(65,79,73) = \widehat{\alpha}(\bar{t_2})$.
   $\widehat{\alpha}(\bar{t_2})^{\widehat{\alpha}(y)} = (1,4,5)(2,17,15)(3,14,16)...(78,97,86)(79,99,87) = \widehat{\alpha}(t_1)$.
   $\widehat{\alpha}(t_2)^{\widehat{\alpha}(y)} = (1,5,4)(2,15,17)(3,16,14)...(78,86,97)(79,87,99) = \widehat{\alpha}(\bar{t_1})$.
   $\widehat{\alpha}(\bar{t_1})^{\widehat{\alpha}(y)} = (1,9,6)(2,24,27)(3,25,22)...(64,72,76)(65,73,79) = \widehat{\alpha}(t_2)$.

2. $\widehat{\alpha}(N)$ acts as N on $\{\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(t_2), \widehat{\alpha}(\bar{t_0}), \widehat{\alpha}(\bar{t_1}), \widehat{\alpha}(\bar{t_2})\}$ by conjugation. We note that
   $\widehat{\alpha}(x) = \{(\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(\bar{t_0}), \widehat{\alpha}(\bar{t_1}))(\widehat{\alpha}(t_2), \widehat{\alpha}(\bar{t_2})\}$,
   $\widehat{\alpha}(y) = \{(\widehat{\alpha}(t_1), \widehat{\alpha}(\bar{t_2}))(\widehat{\alpha}(t_1), \widehat{\alpha}(t_2))\}$ and $< \widehat{\alpha}(x), \widehat{\alpha}(y) > \cong N = S_4$.
   Then, $\widehat{\alpha}(G) = \widehat{\alpha}(3^{*3} : S_4) = \widehat{\alpha} < N, t_0 > = < \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$ is a homomorphic

image of $(3^{*3} : S_4)$.
Next, we want the homomorphic image of $G = \frac{3^{*3}:S_4}{[(1,\bar{2})(\bar{1},2)t_1]^5,[(1,0,2)(\bar{1},\bar{0},\bar{2})t_0t_1t_2=\bar{t_0}\bar{t_2}\bar{t_1}\bar{t_0}]}$.
Now, $<\widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0)> \leq S_{105}$ is a homomorphic image of
$G = \frac{3^{*3}:S_4}{[(1,\bar{2})(\bar{1},2)t_1]^5,[(1,0,2)(\bar{1},\bar{0},\bar{2})t_0t_1t_2=\bar{t_0}\bar{t_2}\bar{t_1}\bar{t_0}]}$ if the additional relations
$(1,\bar{2})(\bar{1},2)t_1\bar{t_2}t_1 = \bar{t_1}t_2$ and $(1,0,2)(\bar{1},\bar{0},\bar{2})t_0t_1t_2 = \bar{t_0}\bar{t_2}\bar{t_1}\bar{t_0}$ hold in
$<\widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0)>$.

First, we want verify the relation-I which is $\widehat{\alpha}((1,\bar{2})(\bar{1},2)t_1\bar{t_2}t_1) = \widehat{\alpha}(\bar{t_1}t_2)$.
On the left side of our equation
$\widehat{\alpha}((1,\bar{2})(\bar{1},2)t_1\bar{t_2}t_1) = \widehat{\alpha}((1,\bar{2})(\bar{1},2))\widehat{\alpha}(t_1)\widehat{\alpha}(\bar{t_2})\widehat{\alpha}(t_1)$
$= (4,6)(5,9)(7,12)(8,13)(10,18)(11,19)...(90,100)(94,103)(101,102)$
$(1,4,5)(2,17,15)(3,14,16)...(78,97,86)(79,99,87)$
$(1,6,9)(2,27,24)(3,22,25)...(64,76,72)(65,79,73)$
$(1,4,5)(2,17,15)(3,14,16)...(78,97,86)(79,99,87)$
$= (1,33,4,9,28)(2,92,17,24,72)...(63,73,77,87,93)(64,74,76,86,91)$.

On the right side of our equation
$\widehat{\alpha}(\bar{t_1}t_2) = \widehat{\alpha}(\bar{t_1})\widehat{\alpha}(t_2)$
$= (1,5,4)(2,15,17)(3,16,14)...(78,86,97)(79,87,99)$
$(1,9,6)(2,24,27)(3,25,22)...(64,72,76)(65,73,79)$
$= (1,33,4,9,28)(2,92,17,24,72)...(63,73,77,87,93)(64,74,76,86,91)$.

Second, we want to verify the relation-II which is
$\widehat{\alpha}((1,0,2)(\bar{1},\bar{0},\bar{2})t_0t_1t_2) = \widehat{\alpha}(\bar{t_0}\bar{t_1}\bar{t_2}\bar{t_0})$.

On the left side of our equation
$\widehat{\alpha}((1,0,2)(\bar{1},\bar{0},\bar{2})t_0t_1t_2) = \widehat{\alpha}((1,0,2)(\bar{1},\bar{0},\bar{2}))\widehat{\alpha}(t_0)\widehat{\alpha}(t_1)\widehat{\alpha}(t_2)$
$= (2,9,4)(3,6,5)(7,24,20)(8,27,21)...(78,84,96)(88,103,90)(94,100,101)$
$(1,2,3)(4,7,8)(5,10,11)(6,12,13)...(75,103,92)(80,104,82)(83,105,85)$
$(1,4,5)(2,17,15)(3,14,16)(6,28,29)...(77,93,89)(78,97,86)(79,99,87)$
$(1,9,6)(2,24,27)(3,25,22)(4,37,38)...(63,75,77)(64,72,76)(65,73,79)$
$= (1,97,12,35,50,8,73)(2,43,18,11,54,3,102)...(46,58,85,71,87,95,62)$.

On the right side of our equation

$\widehat{\alpha}(\bar{t_0}\bar{t_1}\bar{t_2}\bar{t_0}) = \widehat{\alpha}(\bar{t_0})\widehat{\alpha}(\bar{t_2})\widehat{\alpha}(\bar{t_1})\widehat{\alpha}(\bar{t_0})$
$= (1,3,2)(4,8,7)(5,11,10)(6,13,12)...(75,92,103)(80,82,104)(83,85,105)$
$(1,6,9)(2,27,24)(3,22,25)(4,38,37)...(63,77,75)(64,76,72)(65,79,73)$
$(1,5,4)(2,15,17)(3,16,14)(6,29,28)...(77,89,93)(78,86,97)(79,87,99)$
$(1,3,2)(4,8,7)(5,11,10)(6,13,12)...(75,92,103)(80,82,104)(83,85,105)$
$= (1,97,12,35,50,8,73)(2,43,18,11,54,3,102)...(46,58,85,71,87,95,62).$

Therefore, $\widehat{\alpha} : G \rightarrow S_{105}$ is a homomorphism map.
Now by First Isomorphism Theorem, we have
$G|ker(\widehat{\alpha}) \cong im(\widehat{\alpha})$
$\Rightarrow \frac{|G|}{|ker(\widehat{\alpha})|} = |im(\widehat{\alpha})| = | < \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0)| = 2,520$
$\Rightarrow |G| = 2,520 \times |ker(\widehat{\alpha})|$, where $(|ker(\widehat{\alpha})| \geq 1)$.
$\Rightarrow |G| \geq 2,520.$
But early we found that $|G| \leq 2,520$. Therefore, we can conclude that $|G| = 2,520$. Since $|G| = 2,520$,
$\Rightarrow |ker(\widehat{\alpha})| = 1.$
$\Rightarrow \widehat{\alpha}$ is also an isomorphism.
$\Rightarrow G \cong \widehat{\alpha}(G) =< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >.$

It is clear from the Cayley diagram of G over S_4 that every element of G can be written as πw, where π is a permutation of S_4 on six letters, and w is a word in the t_{i_s} of the length at most four. We call this representation of elements of G the symmetric representation. We now give examples of elements of G in both symmetric and permutation representation.

Permutation Representation to Symmetric Presentation

Let $p = (1,8,47,44,34,99,24)(2,26,69,12,10,84,25)(3,15,88,56,42,82,9)$
$(4,41,104,31,93,97,39)(5,32,63,76,90,37,13)(6,11,59,65,78,101,68)$
$(7,70,83,18,14,92,36)(16,86,94,54,46,33,19)(17,100,96,40,85,81,22)$
$(20,79,74,95,38,43,105)(21,77,64,52,57,29,45)(23,66,49,50,58,89,27)$
$(28,51,60,87,91,102,98)(30,35,73,62,53,55,48)(61,75,72,103,71,67,80).$
be an element in Permutation Representation.
We know that $Np = 1^p$
$\Rightarrow Np = p(1) = 8$ and the Table-5 gives that $Nt_1\bar{t_0}$ is labeled 8.

In order the represent p in the symmetric representation form, we must find $n \in N \cong S_4$. We have

$n = pw^{-1}$.

$Np = Nt_1\bar{t_0}$. (An element that labeling (8)).

$p = nw$

$\Rightarrow n = pw^{-1}$

$= p(t_1\bar{t_0})^{-1}$

$\Rightarrow \widehat{\alpha}(n) = p\widehat{\alpha}(t_0)\widehat{\alpha}(\bar{t_1})$

$= (4,6)(5,9)(7,12)(8,13)(10,18)(11,19)(14,22)(15,24)(16,25)(17,27)(20,30)(21,33)$
$(23,36)(26,39)(28,38)(29,37)(31,44)(32,45)(34,50)(35,51)(40,56)(41,57)(42,54)$
$(43,55)(46,66)(47,68)(48,69)(49,71)(58,80)(59,82)(60,83)(61,85)(62,86)(63,89)$
$(64,92)(65,87)(67,96)(70,98)(72,91)(73,99)(74,97)(75,93)(81,104)(84,105)(88,95)$
$(90,100)(94,103)(101,102)$

From the labeling table we have,

$n = (1,\bar{2})(\bar{1},2)$.

Therefore, p can be written in the symmetric representation as

$p = (1,\bar{2})(\bar{1},2)t_1\bar{t_0}$.

Symmetric Presentation to Permutation Representation

Let $\{(1,0,2)(\bar{1},\bar{0},\bar{2})t_0t_1t_2\}$ be an element in the symmetric representation.

$\Rightarrow p = \widehat{\alpha}((1,0,2)(\bar{1},\bar{0},\bar{2}))\widehat{\alpha}(t_0)\widehat{\alpha}(t_1)\widehat{\alpha}(t_2)$

$= (1,97,12,35,50,8,73)(2,43,18,11,54,3,102)(4,99,36,91,40,32,105)$
$(5,37,25,94,78,41,59)(6,81,44,57,75,26,74)(7,77,39,17,42,19,80)$
$(9,83,56,79,100,15,29)(10,55,22,23,76,13,61)(14,101,27,67,31,45,98)$
$(16,52,24,49,34,51,66)(20,82,96,93,103,86,48)(21,33,69,63,53,92,47)$
$(28,84,104,38,89,88,64)(30,68,65,90,72,70,60)(46,58,85,71,87,95,62)$.

Therefore, $\{(1,0,2)(\bar{1},\bar{0},\bar{2})t_0t_1t_2\}$ can be written in the permutation representation as

$(1,0,2)(\bar{1},\bar{0},\bar{2})t_0t_1t_2 =$

$(1,97,12,35,50,8,73)(2,43,18,11,54,3,102)(4,99,36,91,40,32,105)$
$(5,37,25,94,78,41,59)(6,81,44,57,75,26,74)(7,77,39,17,42,19,80)$
$(9,83,56,79,100,15,29)(10,55,22,23,76,13,61)(14,101,27,67,31,45,98)$
$(16,52,24,49,34,51,66)(20,82,96,93,103,86,48)(21,33,69,63,53,92,47)$

$(28, 84, 104, 38, 89, 88, 64)(30, 68, 65, 90, 72, 70, 60)(46, 58, 85, 71, 87, 95, 62).$

7.2 Lemmas

Lemma-1

Proof. $010 = (0,1)(\bar{0},\bar{1})\bar{0}\bar{1} \in [01].$ □

Lemma-2

Proof. $0\underline{120} = 0(2,0,1)(\bar{2},\bar{0},\bar{1})\bar{1}\bar{0}\bar{2}\bar{1} = (2,0,1)(\bar{2},\bar{0},\bar{1})1\bar{1}\bar{0}\bar{2}\bar{1}$
$= (2,0,1)(\bar{2},\bar{0},\bar{1})\bar{0}\bar{2}\bar{1} \in [01\bar{2}].$ □

Lemma-3

Proof. $0\underline{121} = 0(2,1)(\bar{2},\bar{1})\bar{1}\bar{2} = (2,1)(\bar{2},\bar{1})0\bar{1}\bar{2} \in [012].$ □

Lemma-4

Proof. $01\underline{\bar{0}2} = 01(\bar{0},2)(\bar{2},0)0\bar{2}0 = (\bar{0},2)(\bar{2},0)\underline{\bar{2}10\bar{2}}0 = (\bar{0},2)(\bar{2},0)(\bar{0},2,\bar{1})(0,\bar{2},1)2\bar{0}\bar{1}0$
$= (0,1)(\bar{0},\bar{1})2\bar{0}\bar{1}0 \in [012\bar{1}].$ □

Lemma-5

Proof. $\underline{01\bar{2}0} = (2,\bar{0},\bar{1})(\bar{2},0,1)\bar{0}2\bar{1} \in [012].$ □

Lemma-6

Proof. $0\underline{1\bar{2}}1 = 0(\bar{2},1)(\bar{1},2)\bar{1}2\bar{1}1 = 0(\bar{2},1)(\bar{1},2)0\bar{1}2 \in [01\bar{2}].$ □

Lemma-7

Proof. $\underline{01}\bar{2}\bar{0} = (0,1)(\bar{0},\bar{1})\bar{0}\bar{1}\underline{\bar{0}\bar{2}\bar{0}} = (0,1)(\bar{0},\bar{1})\bar{0}\bar{1}(0,2)(\bar{0},\bar{2})02$
$= (0,1,2)(\bar{0},\bar{1},\bar{2})\bar{2}\bar{1}02 \in [012\bar{0}].$ □

Lemma-8

Proof. $01\bar{2}\bar{1} = (0,1)(\bar{0},\bar{1})\bar{0}\underline{\bar{1}\bar{0}\bar{2}\bar{1}} = (0,1)(\bar{0},\bar{1})\bar{0}(2,1,0)(\bar{2},\bar{1},\bar{0})120$
$= (1,2)(\bar{1},\bar{2})\bar{2}120 \in [01\bar{0}\bar{2}].$ □

Lemma-9

Proof. $\underline{012}\bar{1}0 = (0,1,2)(\bar{0},\bar{1},\bar{2})\bar{0}\bar{2}\underline{\bar{1}\bar{0}\bar{1}}0 = (0,1,2)(\bar{0},\bar{1},\bar{2})\bar{0}\bar{2}(0,1)(\bar{0},\bar{1})100$
$= (1,2)(\bar{1},\bar{2})\bar{1}\bar{2}1\bar{0} \in [01\bar{0}\bar{2}]$. □

Lemma-10

Proof. $\underline{012}\bar{1}\bar{0} = (0,1,2)(\bar{0},\bar{1},\bar{2})\bar{0}\bar{2}\underline{\bar{1}\bar{0}\bar{1}}\bar{0} = (0,1,2)(\bar{0},\bar{1},\bar{2})\bar{0}\bar{2}(0,1)(\bar{0},\bar{1})10\bar{0}$
$= (1,2)(\bar{1},\bar{2})\bar{1}\bar{2}1 \in [01\bar{0}]$. □

Lemma-11

Proof. $01\bar{0}\bar{2}1 = 01(0,2)(\bar{0},\bar{2})0201 = (0,2)(\bar{0},\bar{2})\underline{210}201$
$= (0,2)(\bar{0},\bar{2})(2,1,0)(\bar{2},\bar{1},\bar{0})\bar{2}\bar{0}\bar{1}\bar{2}201 = (0,1)(\bar{0},\bar{1})\bar{2}\underline{\bar{0}\bar{1}}01 = (0,1)(\bar{0},\bar{1})\bar{2}(0,1)(\bar{0},\bar{1})01001$
$= \bar{2}0\underline{1\bar{0}1} = \bar{2}0(\bar{0},1)(\bar{1},0)\bar{1}0 = (\bar{0},1)(\bar{1},0)\bar{2}\bar{1}\bar{1}0 = (\bar{0},1)(\bar{1},0)\bar{2}10 \in [01\bar{2}]$. □

Chapter 8

Construction of $3{\cdot}A_7$ Using Different Approach

Now $L_2(7) \times 3$ is a maximal subgroup of $3{\cdot}A_7$ containing S_4. In order to perform the double coset enumeration of G over S_4, we will perform a double coset enumeration of

$$G = \frac{3^{*3} :_m S_4}{[(1,\bar{2})(\bar{1},2)t_1]^5} \cong 3{\cdot}A_7$$

over $H = L_2(7) \times 3 \cong < N, (0,2)(\bar{0},\bar{2})t_0\bar{t_2}\bar{t_0}\bar{t_1} >$ and a double coset enumeration of H over N. Factoring the progenitor $3^{*3} :_m S_4$ by the relations $[(1,\bar{2})(\bar{1},2)t_1]^5$ and $(0,2)(\bar{0},\bar{2})t_0\bar{t_2}\bar{t_0}\bar{t_1} = 1$, we obtain the finite image

$$G \cong \frac{3^{*3} :_m S_4}{[(1,\bar{2})(\bar{1},2)t_1]^5}$$

Symmetric presentation of the progenitor $3^{*3} :_m S_4$ can be given by:

$$3^{*3} :_m S_4 \cong < x, y, t | x^4, y^2, (yx)^3, t^3, [ty], (x^2t)^2 >$$

The control group is $N = S_4 \cong < x, y | x^4 = y^2 = (yx)^3 >$. $H =< N, (0,2)(\bar{0},\bar{2})t_0\bar{t_2}\bar{t_0}\bar{t_1} >$ and every single coset of H contains 21 single cosets of G. The action on the symmetric generators is given by $x \sim (0,1,\bar{0},\bar{1})(2,\bar{2})$, $y \sim (1,\bar{2})(2,\bar{1})$. The relations are $[(1,\bar{2})(\bar{1},2)t_1]^5$ and $(0,2)(\bar{0},\bar{2})t_0\bar{t_2} = t_1t_0$.

First, we study our relations.

Relation-I

Let $\pi = (1,\bar{2})(2,\bar{1})$, then $(\pi t_1)^5 = 1$

$$\begin{aligned}
(\pi t_1)^5 &= 1\\
\pi t_1 \pi t_1 \pi t_1 \pi t_1 \pi t_1 &= 1\\
\pi t_1 \pi t_1 \pi t_1 \pi \pi \pi^{-1} t_1 \pi t_1 &= 1\\
\pi t_1 \pi t_1 \pi t_1 \pi^2 (t_1)^{\pi} t_1 &= 1\\
\pi t_1 \pi t_1 \pi \pi^2 \pi^{-2} t_1 \pi^2 (t_1)^{\pi} t_1 &= 1\\
\pi t_1 \pi \pi^3 \pi^{-3} t_1 \pi^3 (t_1)^{\pi^2} (t_1)^{\pi} t_1 &= 1\\
\pi t_1 \pi^4 (t_1)^{\pi^3} (t_1)^{\pi^2} (t_1)^{\pi} t_1 &= 1\\
\pi \pi^4 \pi^{-4} t_1 \pi^4 (t_1)^{\pi^3} (t_1)^{\pi^2} (t_1)^{\pi} t_1 &= 1\\
\pi^5 (t_1)^{\pi^4} (t_1)^{\pi^3} (t_1)^{\pi^2} (t_1)^{\pi} t_1 &= 1
\end{aligned}$$

Since, $\pi = (1,\bar{2})(2,\bar{1})$
$\Rightarrow \pi^2 = (1,\bar{2})(2,\bar{1})(1,\bar{2})(2,\bar{1}) = 1.$
$\Rightarrow \pi^3 = (1,\bar{2})(2,\bar{1}).$
$\Rightarrow \pi^4 = 1.$
$\Rightarrow \pi^5 = (1,\bar{2})(2,\bar{1}).$

$$\begin{aligned}
\pi^5 (t_1)^{\pi^4} (t_1)^{\pi^3} (t_1)^{\pi^2} (t_1)^{\pi} t_1 &= 1\\
\Leftrightarrow \pi^5 t_1 \bar{t_2} t_1 \bar{t_2} t_1 &= 1\\
\Leftrightarrow \pi^5 t_1 \bar{t_2} t_1 &= \bar{t_1} t_2
\end{aligned}$$

Relation-II

$(0,2)(\bar{0},\bar{2}) t_0 \bar{t_2} = t_1 t_0.$

Double coset enumeration of G over S_4.

We note that,

$$HeN = \{Hen | n \in N\} = \{Hn | n \in H\} = \{H\}.$$

Let [*] represents for the double coset HeN. [*] has only one distinct right coset, which is the identity single coset. Since, $N \cong S_4$, it is transitive on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$. Therefore, [*] has a single orbit namely $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$. Now, we take a representative from the orbit called 0 and find which double coset Ht_0 belongs.

Clearly, $Ht_0 \in Ht_0N = \{Ht_0^n | n \in N\} = \{Ht_0, Ht_1, Ht_2, Ht_{\bar{0}}, Ht_{\bar{1}}, Ht_{\bar{2}}\}$ and let us denote Ht_0N by [0]. Now, consider the coset stabilizer $N^{(0)}$. The coset stabilizer $N^{(0)}$ is equal to the point stabilizer N^0, it is given by:

$$N^{(0)} = N^0 = \{e, (1, \bar{2})(2, \bar{1}), (1, 2)(\bar{1}, \bar{2}), (1, \bar{1})(2, \bar{2})\}.$$

Therefore, the number of single cosets in [0] are

$$\frac{|N|}{|N^{(0)}|} = \frac{4!}{4} = \frac{24}{4} = 6.$$

The orbits of $N^{(0)}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}$, $\{\bar{0}\}$ and $\{1, 2, \bar{1}, \bar{2}\}$. Now, take one representative from each orbit, namely 0, $\bar{0}$ and 1, then find the double coset to which Ht_0t_0, $Ht_0t_{\bar{0}}$ and Ht_0t_1 belong? However,

$Ht_0t_0 = Ht_{\bar{0}} \in [0]$
$Ht_0t_{\bar{0}} = He \in [*]$
$Ht_0t_1 = Ht_0t_1 \in Ht_0t_1N$ (New double coset denoted by [01])

Therefore, one symmetric generator goes back to the identity double coset [*]; one stays in [0]; four extend to new double coset [01].

Next, consider the new double coset [01].

By conjugate our relation-2 $(0, 2)(\bar{0}, \bar{2})t_0t_{\bar{2}} = t_1t_0$ with the elements of the control group N, we obtain the following relations.

- $((0, 2)(\bar{0}, \bar{2})t_0t_{\bar{2}})^{\{(0,1)(\bar{0},\bar{1})\}} = (t_1t_0)^{\{(0,1)(\bar{0},\bar{1})\}}$
  $\Rightarrow (2, 1)(\bar{2}, \bar{1})t_1t_{\bar{2}} = t_0t_1.$

- $((0,2)(\bar{0},\bar{2})t_0\bar{t_2})^{\{(2,\bar{0})(\bar{2},0)\}} = (t_1t_0)^{\{(2,\bar{0})(\bar{2},0)\}}$
  $\Rightarrow (2,0)(\bar{2},\bar{0})\bar{t_2}t_0 = t_1\bar{t_2}$.

Since $N^{(01)} \geq N^{01} =< e >$ and
$H(t_0t_1)^{\{(1,\bar{2},0)(2,\bar{0},\bar{1})\}} = Ht_1\bar{t_2} = Ht_0t_1 \Rightarrow (1,\bar{2},0)(2,\bar{0},\bar{1}) \in N^{(01)}$
$H(t_0t_1)^{\{(1,0,\bar{2})(\bar{1},\bar{0},2)\}} = H\bar{t_2}t_0 = Ht_1\bar{t_2} = Ht_0t_1 \Rightarrow (1,0,\bar{2})(\bar{1},\bar{0},2) \in N^{(01)}$
$\Rightarrow N^{(01\bar{0})} =< (1,\bar{2},0)(2,\bar{0},\bar{1}),(1,0,\bar{2})(\bar{1},\bar{0},2) >= \{e,(1,\bar{2},0)(2,\bar{0},\bar{1}),(1,0,\bar{2})(2,\bar{1},\bar{0})\}$.

Therefore, the number of single cosets in the double coset $[01\bar{0}]$ are at most

$\frac{|N|}{|N^{(01)}|} = \frac{4!}{3} = \frac{24}{3} = 8$. Each single coset has three different names.

Now, take (01) conjugates with the elements of $N^{(01\bar{0})}$, we obtain the following relations:

$$01 \sim 1\bar{2} \sim \bar{2}0$$

By conjugating $(01 \sim 1\bar{2} \sim \bar{2}0)$ with the elements of transversal of $N^{(01)}$, we obtain the following single cosets of double coset [01]:

$$\begin{gathered}
01 \sim 1\bar{2} \sim \bar{2}0 \\
1\bar{0} \sim \bar{0}2 \sim 21 \\
0\bar{2} \sim \bar{2}1 \sim 10 \\
\bar{0}\bar{1} \sim \bar{1}\bar{2} \sim \bar{2}\bar{0} \\
12 \sim 2\bar{0} \sim \bar{0}1 \\
\bar{1}0 \sim 02 \sim 2\bar{1} \\
\bar{2}\bar{1} \sim \bar{1}\bar{0} \sim \bar{0}\bar{2} \\
\bar{1}2 \sim 20 \sim 0\bar{1}
\end{gathered}$$

The orbits of $N^{(01)}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{1,0,\bar{2}\}$ and $\{2,\bar{1},\bar{0}\}$. Now, take one representative from each orbit, namely 1 and $\bar{1}$, then find the double coset to which $Ht_0t_1t_1$ and $Ht_0t_1\bar{t_1}$ belong? However,

$Ht_0t_1t_1 = Ht_0\bar{t_1} \in [01]$.
$Ht_0t_1\bar{t_1} = Ht_0 \in [0]$.

Since all symmetric generators go back to the previous double cosets and none of them advance. Thus, we must have completed the double coset enumeration. The information is summarized in the Cayley diagram below.

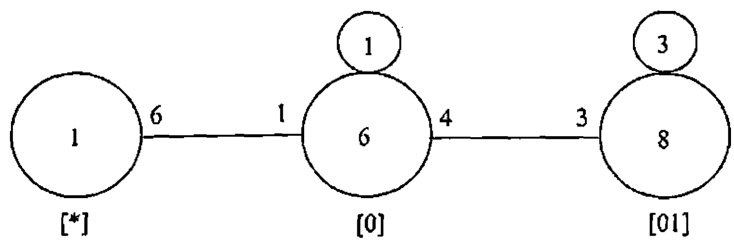

Figure 8.1: The Cayley diagram of $3{\cdot}A_7$ over $H \cong L_2(7) \times 3$.

The double coset enumeration shows that the index of $N \cong S_4$ is at most:
$|G| \leq (15)|H| = (15) \times 504 = 7,560$.

The homomorphic image of G.

Since every single coset of H contains 21 single cosets of G. Thus, the total number of single cosets in G must be

$$15 \times 21 = 315.$$

Define $\widehat{\alpha} : G \to S_{315}$, where (315) is the total number of single cosets gives the action of G on the 315 cosets. From the permutation table, we compute the action of x, y and t_0 on the 315 cosets as follow:
$\widehat{\alpha}(x) = (2,4,3,5)(6,9)(7,14,11,15)...(306,310,307,311)(308,312,309,313)$.
$\widehat{\alpha}(y) = (4,6)(5,9)(7,12)(8,13)(10,18)(11,19)...(299,307)(301,308)(305,309)$.
$\widehat{\alpha}(t_0) = (1,2,3)(4,7,8)(5,10,11)...(310,314,311)(312,315,313)$.
In order to verify that $\widehat{\alpha}$ is a homomorphism, we required

1. We want to show that $\widehat{\alpha}(t_0)$ has exactly 6 conjugates under conjugation by $< \widehat{\alpha}(x), \widehat{\alpha}(y) >$.
   $\widehat{\alpha}(t_0)^{\widehat{\alpha}(x)} = (1,4,5)(2,17,15)(3,14,16)...(306,307,314)(308,309,315) = \widehat{\alpha}(t_1)$.
   $\widehat{\alpha}(t_1)^{\widehat{\alpha}(x)} = (1,3,2)(4,8,7)(5,11,10)...(310,311,314)(312,313,315) = \widehat{\alpha}(\bar{t_0})$.

$\widehat{\alpha}(\bar{t_0})^{\widehat{\alpha}(x)} = (1,5,4)(2,15,17)(3,16,14)...(306,314,307)(308,315,309) = \widehat{\alpha}(\bar{t_1})$.
$\widehat{\alpha}(\bar{t_1})^{\widehat{\alpha}(x)} = (1,2,3)(4,7,8)(5,10,11)...(310,314,311)(312,315,313) = \widehat{\alpha}(t_0)$.
$\widehat{\alpha}(t_2)^{\widehat{\alpha}(x)} = (1,6,9)(2,27,24)(3,22,25)...(295,299,314)(301,305,315) = \widehat{\alpha}(\bar{t_2})$.
$\widehat{\alpha}(\bar{t_2})^{\widehat{\alpha}(x)} = (1,9,6)(2,24,27)(3,25,22)...(295,314,299)(301,315,305) = \widehat{\alpha}(t_2)$.

$\widehat{\alpha}(t_1)^{\widehat{\alpha}(y)} = (1,6,9)(2,27,24)(3,22,25)...(295,299,314)(301,305,315) = \widehat{\alpha}(\bar{t_2})$.
$\widehat{\alpha}(\bar{t_2})^{\widehat{\alpha}(y)} = (1,4,5)(2,17,15)(3,14,16)...(306,307,314)(308,309,315) = \widehat{\alpha}(t_1)$.
$\widehat{\alpha}(t_2)^{\widehat{\alpha}(y)} = (1,5,4)(2,15,17)(3,16,14)...(306,314,307)(308,315,309) = \widehat{\alpha}(\bar{t_1})$.
$\widehat{\alpha}(\bar{t_1})^{\widehat{\alpha}(y)} = (1,9,6)(2,24,27)(3,25,22)...(295,314,299)(301,315,305) = \widehat{\alpha}(t_2)$.

2. $\widehat{\alpha}(N)$ acts as N on $\{\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(t_2), \widehat{\alpha}(\bar{t_0}), \widehat{\alpha}(\bar{t_1}), \widehat{\alpha}(\bar{t_2})\}$ by conjugation, that is, if
$x = (0,1,\bar{0},\bar{1})(2,\bar{2})$, and
$y = (1,\bar{2})(2,\bar{1})$, then
$\widehat{\alpha}(x) = \{(\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(\bar{t_0}), \widehat{\alpha}(\bar{t_1}))(\widehat{\alpha}(t_2), \widehat{\alpha}(\bar{t_2})\}$,
$\widehat{\alpha}(y) = \{(\widehat{\alpha}(t_1), \widehat{\alpha}(\bar{t_2}))(\widehat{\alpha}(t_1), \widehat{\alpha}(t_2))\}$.
Then, $\widehat{\alpha}(G) = \widehat{\alpha}(3^{*3} : S_4) = \widehat{\alpha} < N, t_0 >=< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$ is a homomorphic image of $(3^{*3} : S_4)$.
Next, we want the homomorphic image of $G = \frac{3^{*3}:S_4}{[(1,\bar{2})(\bar{1},2)t_1]^5}$.
Since, $< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) > \leq S_{315}$ is a homomorphic image of
$G = \frac{3^{*3}:S_4}{[(1,\bar{2})(\bar{1},2)t_1]^5}$ if the additional relation
$(1,\bar{2})(\bar{1},2)t_1\bar{t_2}t_1 = \bar{t_1}t_2$ holds in $< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$.
Now, we want to show that $\widehat{\alpha}((1,\bar{2})(\bar{1},2)t_1\bar{t_2}t_1) = \widehat{\alpha}(\bar{t_1}t_2)$.
On the left side of our equation
$\widehat{\alpha}((1,\bar{2})(\bar{1},2)t_1\bar{t_2}t_1) = \widehat{\alpha}((1,\bar{2})(\bar{1},2))\widehat{\alpha}(t_1)\widehat{\alpha}(\bar{t_2})\widehat{\alpha}(t_1)$
$= (4,6)(5,9)(7,12)(8,13)(10,18)(11,19)...(299,307)(301,308)(305,309)$
$(1,4,5)(2,17,15)(3,14,16)...(306,307,314)(308,309,315)$
$(1,6,9)(2,27,24)(3,22,25)...(295,299,314)(301,305,315)$
$(1,4,5)(2,17,15)(3,14,16)...(306,307,314)(308,309,315)$
$= (1,33,4,9,28)(2,92,17,24,72)...(224,278,255,262,259)(228,290,234,266,237)$.

On the right side of our equation
$\widehat{\alpha}(\bar{t_1}t_2) = \widehat{\alpha}(\bar{t_1})\widehat{\alpha}(t_2)$

$= (1,5,4)(2,15,17)(3,16,14)...(306,314,307)(308,315,309)$
$(1,9,6)(2,24,27)(3,25,22)...(295,314,299)(301,315,305)$
$= (1,33,4,9,28)(2,92,17,24,72)...(224,278,255,262,259)(228,290,234,266,237).$

Since, $\widehat{\alpha}: G \to S_{315}$ is a homomorphism map. By First Isomorphism Theorem, we have
$G|ker(\widehat{\alpha}) \cong im(\widehat{\alpha})$
$\Rightarrow \frac{|G|}{|ker(\widehat{\alpha})|} = |im(\widehat{\alpha})| = | < \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0)| = 7,560$
$\Rightarrow |G| = 7,560 \times |ker(\widehat{\alpha})|$, where $(|ker(\widehat{\alpha})| \geq 1)$.
$\Rightarrow |G| \geq 7,560.$
But early we found that $|G| \leq 7,560$. Therefore, we can conclude that $|G| = 7,560$. Since
$|G| = 7,560,$
$\Rightarrow |ker(\widehat{\alpha})| = 1.$
$\Rightarrow \widehat{\alpha}$ is also an isomorphism.
$\Rightarrow G \cong \widehat{\alpha}(G) =< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >.$

It is clear from the Cayley diagram of G over S_4 that every element of G can be written as πw, where π is a permutation of S_4 on six letters, and w is a word in the t_{i_s} of the length at most seven. We call this representation of elements of G the symmetric representation. We now give examples of elements of G in both symmetric and permutation representation.

Permutation Representation to Symmetric Presentation

Let $p = (1,15)(2,26)(3,8)(4,16)(6,47)(7,14)(9,59)(12,70)(13,41)(17,23)(18,84)$
$(19,32)(20,60)(22,121)(24,105)(25,143)(27,88)...(266,288)(267,313)(269,279)$
$(273,295)(274,314)(275,306)(281,305)(284,296)(289,300)(292,308)(293,315)$
$(294,310)(302,312).$
be an element in permutation representation.
We know that $Np = 1^p$
$\Rightarrow Np = p(1) = 15$ and the labeling table gives that $Nt_0\bar{t_1}$ is labeled 15.
In order the represent p in the symmetric representation form, we must find $n \in N \cong S_4$.
We have
$Np = 1^p = p(1) = Nt_0\bar{t_1}$. (An element that labeling (15)).
$p = nw$

$\Rightarrow n = pw^{-1}$

$\Rightarrow n = pt_1\bar{t_0}$

$\Rightarrow \widehat{\alpha}(n) = p\widehat{\alpha}(t_1)\widehat{\alpha}(\bar{t_0})$

$= (2,4)(3,5)(7,17)(8,15)(10,14)(11,16)(12,28)(13,29)(18,20)(19,21)(22,30)...$

$(284,298)(285,296)(288,300)(289,303)(292,304)(293,302)(306,311)(307,310)$

$(308,313)(309,312)$

From our multiplication table we have,

$n = (1,0)(\bar{1},\bar{0})$.

Therefore, p can be written in the symmetric representation as

$p = (1,0)(\bar{1},\bar{0})t_0\bar{t_1}$.

Symmetric Presentation to Permutation Representation

Let $\{(1,\bar{2})(\bar{1},2)t_1\bar{t_2}t_1\}$ be an element in the symmetric representation.

$\Rightarrow p = \widehat{\alpha}((1,\bar{2})(\bar{1},2))\widehat{\alpha}(t_1)\widehat{\alpha}(\bar{t_2})\widehat{\alpha}(t_1)$

$= (4,6)(5,9)(7,12)(8,13)(10,18)(11,19)...(299,307)(301,308)(305,309)$

$(1,4,5)(2,17,15)(3,14,16)...(306,307,314)(308,309,315)$

$(1,6,9)(2,27,24)(3,22,25)...(295,299,314)(301,305,315)$

$(1,4,5)(2,17,15)(3,14,16)...(306,307,314)(308,309,315)$

$= (1,33,4,9,28)(2,92,17,24,72)(3,89,14,25,75)(5,37,20,6,53)(7,148,49,39,160)$

$(8,147,46,36,159)(10,69,61,13,156)(11,68,58,12,163)(15,99,62,27,131)$

$(16,103,65,22,136)(18,212,23,82,115)(19,185,26,83,112)(21,29,52,30,38)$

$(31,264,94,117,206)(32,268,90,116,199)(34,165,109,45,242)(35,155,106,44,252)$

$(40,267,126,101,291)(41,263,120,98,279)(42,128,144,57,276)(43,119,142,56,281)$

$(47,157,150,71,193)(48,162,169,66,202)(50,302,81,179,226)(51,296,84,181,222)$

$(54,256,67,139,225)(55,231,70,134,229)(59,192,166,85,151)(60,205,153,80,168)$

$(63,73,130,95,100)(64,74,137,86,102)(76,214,91,135,196)(77,191,93,138,209)$

$(78,293,107,129,248)(79,275,108,118,244)(87,217,189,132,176)(88,243,186,164,239)$

$(96,253,219,154,250)(97,188,216,141,171)(104,289,260,124,312)$

$(105,297,258,180,271)(110,303,236,178,287)(111,285,232,123,311)$

$(113,161,182,149,152)(114,158,213,146,167)(121,277,270,125,298)$

$(122,304,127,283,286)(133,184,143,240,280)(140,220,145,247,273)$

$(170,183,187,241,238)(172,292,197,215,249)(173,307,198,230,295)$

$(174,309,207,257,301)(175,274,208,190,245)(177,221,218,246,251)$

$(194, 288, 261, 210, 313)(195, 310, 211, 284, 233)(200, 300, 269, 282, 227)$
$(201, 306, 299, 235, 314)(203, 308, 305, 254, 315)(204, 294, 265, 272, 223)$
$(224, 278, 255, 262, 259)(228, 290, 234, 266, 237)$.

Therefore, $\{(1, \bar{2})(\bar{1}, 2)t_1\bar{t_2}t_1\}$ can be written in the permutation representation as follows
$(1, \bar{2})(\bar{1}, 2)t_1\bar{t_2}t_1$
$= (1, 33, 4, 9, 28)(2, 92, 17, 24, 72)(3, 89, 14, 25, 75)(5, 37, 20, 6, 53)(7, 148, 49, 39, 160)$
$(8, 147, 46, 36, 159)(10, 69, 61, 13, 156)(11, 68, 58, 12, 163)(15, 99, 62, 27, 131)$
$(16, 103, 65, 22, 136)(18, 212, 23, 82, 115)(19, 185, 26, 83, 112)(21, 29, 52, 30, 38)$
$(31, 264, 94, 117, 206)(32, 268, 90, 116, 199)(34, 165, 109, 45, 242)(35, 155, 106, 44, 252)$
$(40, 267, 126, 101, 291)(41, 263, 120, 98, 279)(42, 128, 144, 57, 276)(43, 119, 142, 56, 281)$
$(47, 157, 150, 71, 193)(48, 162, 169, 66, 202)(50, 302, 81, 179, 226)(51, 296, 84, 181, 222)$
$(54, 256, 67, 139, 225)(55, 231, 70, 134, 229)(59, 192, 166, 85, 151)(60, 205, 153, 80, 168)$
$(63, 73, 130, 95, 100)(64, 74, 137, 86, 102)(76, 214, 91, 135, 196)(77, 191, 93, 138, 209)$
$(78, 293, 107, 129, 248)(79, 275, 108, 118, 244)(87, 217, 189, 132, 176)(88, 243, 186, 164, 239)$
$(96, 253, 219, 154, 250)(97, 188, 216, 141, 171)(104, 289, 260, 124, 312)$
$(105, 297, 258, 180, 271)(110, 303, 236, 178, 287)(111, 285, 232, 123, 311)$
$(113, 161, 182, 149, 152)(114, 158, 213, 146, 167)(121, 277, 270, 125, 298)$
$(122, 304, 127, 283, 286)(133, 184, 143, 240, 280)(140, 220, 145, 247, 273)$
$(170, 183, 187, 241, 238)(172, 292, 197, 215, 249)(173, 307, 198, 230, 295)$
$(174, 309, 207, 257, 301)(175, 274, 208, 190, 245)(177, 221, 218, 246, 251)$
$(194, 288, 261, 210, 313)(195, 310, 211, 284, 233)(200, 300, 269, 282, 227)$
$(201, 306, 299, 235, 314)(203, 308, 305, 254, 315)(204, 294, 265, 272, 223)$
$(224, 278, 255, 262, 259)(228, 290, 234, 266, 237)$.

Chapter 9

Construction of $M_{12}:2$

9.1 Double coset enumeration of G over S_5

Factoring the progenitor $3^{*5} :_m S_5$ by the relations $[(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_0]^4$ and $[(0,1,2,3,4)(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4})t_0]^8$. We obtain the finite image

$$G \cong \frac{3^{*5} :_m S_5}{[(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_0]^4[(0,1,2,3,4)(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4})t_0]^8}$$

The symmetric presentation of the progenitor $3^{*5} :_m S_5$ can be given by:

$$3^{*5} :_m S_5 \cong < x, y, t | x^5, y^2, (xy)^4, t^3, (yt)^2, (yt^x)^2, (yt^{x^2})^2 >$$

The control group is $N = S_5 \cong < x, y | x^5 = y^2 = (xy)^4 >$. The action on the symmetric generators is given by $x \sim (0,1,2,3,4)(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4})$, $y \sim (0,\bar{0})(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})$, and our relations are $[(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})t_0]^4$ and $[(0,1,2,3,4)(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4})t_0]^8$.

First, we study our relations.

Relation-I

Let $\pi = (0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})$, then $(\pi t_0)^4 = 1$

$$\begin{aligned}
(\pi t_0)^4 &= 1 \\
\pi t_0 \pi t_0 \pi t_0 \pi t_0 &= 1 \\
\pi t_0 \pi t_0 \pi \pi \pi^{-1} t_0 \pi t_0 &= 1 \\
\pi t_0 \pi t_0 \pi^2 (t_0)^{\pi} t_0 &= 1 \\
\pi t_0 \pi \pi^2 \pi^{-2} t_0 \pi^2 (t_0)^{\pi} t_0 &= 1 \\
\pi \pi^3 \pi^{-3} t_0 \pi^3 (t_0)^{\pi^2} (t_0)^{\pi} t_0 &= 1 \\
\pi^4 (t_0)^{\pi^3} (t_0)^{\pi^2} (t_0)^{\pi} t_0 &= 1
\end{aligned}$$

Since, $\pi = (0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})$
$\Rightarrow \pi^2 = (0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4})(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4}) = (2,4,3)(\bar{2},\bar{4},\bar{3})$.
$\Rightarrow \pi^3 = (2,4,3)(\bar{2},\bar{4},\bar{3})(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4}) = (0,\bar{1})(\bar{0},1)(2,\bar{2})(3,\bar{3})(4,\bar{4})$.
$\Rightarrow \pi^4 = (0,\bar{1})(\bar{0},1)(2,\bar{2})(3,\bar{3})(4,\bar{4})(0,\bar{1})(\bar{0},1)(2,\bar{3},4,\bar{2},3,\bar{4}) = (2,3,4)(\bar{2},\bar{3},\bar{4})$.

$$\begin{aligned}
\pi^4 (t_0)^{\pi^3} (t_0)^{\pi^2} (t_0)^{\pi} t_0 &= 1 \\
\Leftrightarrow \pi^4 \bar{t_1} t_0 \bar{t_1} t_0 &= 1 \\
\Leftrightarrow \pi^4 \bar{t_1} t_0 &= \bar{t_0} t_1
\end{aligned}$$

Relation-II

Let $\pi = (0,1,2,3,4)(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4})$, then $(\pi t_0)^8 = 1$.

$$\begin{aligned}
(\pi t_0)^8 &= 1\\
\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0 &= 1\\
\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi\pi\pi^{-1}t_0\pi t_0 &= 1\\
\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi^2(t_0)^{\pi}t_0 &= 1\\
\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi\pi^2\pi^{-2}t_0\pi^2(t_0)^{\pi}t_0 &= 1\\
\pi t_0\pi t_0\pi t_0\pi t_0\pi t_0\pi^3(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\pi t_0\pi t_0\pi t_0\pi t_0\pi\pi^3\pi^{-3}t_0\pi^3(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\pi t_0\pi t_0\pi t_0\pi t_0\pi^4(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\pi t_0\pi t_0\pi t_0\pi\pi^4\pi^{-4}t_0\pi^4(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\pi t_0\pi t_0\pi\pi^5\pi^{-5}t_0\pi^5(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\pi t_0\pi t_0\pi^6(t_0)^{\pi^5}(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\pi t_0\pi\pi^6\pi^{-6}t_0\pi^6(t_0)^{\pi^5}(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\pi\pi^7\pi^{-7}t_0\pi^7(t_0)^{\pi^6}(t_0)^{\pi^5}(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\pi^8(t_0)^{\pi^7}(t_0)^{\pi^6}(t_0)^{\pi^5}(t_0)^{\pi^4}(t_0)^{\pi^3}(t_0)^{\pi^2}(t_0)^{\pi}t_0 &= 1\\
\pi^8 t_2t_1t_0t_4 &= \bar{t_0}\bar{t_1}\bar{t_2}\bar{t_3}
\end{aligned}$$

Since, $\pi = (0,1,2,3,4)(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4})$

$\Rightarrow \pi^2 = (0,2,4,1,3)(\bar{0},\bar{2},\bar{4},\bar{1},\bar{3})$

$\Rightarrow \pi^3 = (0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})$

$\Rightarrow \pi^4 = (0,4,3,2,1)(\bar{0},\bar{4},\bar{3},\bar{2},\bar{1})$

$\Rightarrow \pi^5 = 1$

$\Rightarrow \pi^6 = (0,1,2,3,4)(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}) = \pi$

$\Rightarrow \pi^7 = (0,2,4,1,3)(\bar{0},\bar{2},\bar{4},\bar{1},\bar{3}) = \pi^2$

$\Rightarrow \pi^8 = (0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2}) = \pi^3$

Double coset enumeration of G over S_5.

Unless otherwise specified, proof of all the lemmas in Chapter-9 can be found on pages 165-185.

We note that,

$$NeN = \{Nen|n \in N\} = \{Nn|n \in N\} = \{N\}.$$

Let [*] represents the double coset NeN. [*] has only one distinct single coset, namely N. Since, $N \cong S_5$ acts transitively on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$. Therefore, N has one orbit namely $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$. Now, we take one representative 0 from the orbit, and find which double coset Nt_0 belongs.

Clearly, $Nt_0 \in Nt_0N = \{Nt_0^n|n \in N\} = \{Nt_0, Nt_1, Nt_2, Nt_3, Nt_4, N\bar{t_0}, N\bar{t_1}, N\bar{t_2}, N\bar{t_3}, N\bar{t_4}\}$, and denoted Nt_0N by [0]. Now, consider the coset stabilizer $N^{(0)}$. The coset stabilizer $N^{(0)}$ is equal to the point stabilizer N^0 and is given by:

$$N^{(0)} = N^0 = \{Nt_0, Nt_1, Nt_2, Nt_3, Nt_4, N\bar{t_0}, N\bar{t_1}, N\bar{t_2}, N\bar{t_3}, N\bar{t_4}\}.$$

Therefore, the number of single cosets in [0] are at most

$$\frac{|N|}{|N^{(0)}|} = \frac{5!}{12} = \frac{120}{12} = 10.$$

The orbits of $N^{(0)}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{0\}$, $\{\bar{0}\}$,$\{1,2,3,4\}$, and $\{\bar{1},\bar{2},\bar{3},\bar{4}\}$. Now, take one representative from each orbit, namely 0, $\bar{0}$, $\{1\}$ and $\bar{1}$, then find the double coset to which Nt_0t_0, $Nt_0\bar{t_0}$, Nt_0t_1 and $Nt_0\bar{t_1}$ belong. However,

$Nt_0t_0 = N\bar{t_0} \in [0]$
$Nt_0\bar{t_0} = Ne \in [*]$
$Nt_0t_1 = Nt_0t_1 \in Nt_0t_1N$ (New double coset denote by [01])
$Nt_0\bar{t_1} = Nt_0\bar{t_1} \in Nt_0\bar{t_1}N$ (New double coset denote by $[0\bar{1}]$)

Therefore, one symmetric generator goes back to the identity double coset [*]; one stays in [0]; four extend to new double coset [01], and the other four extend to new double coset $[0\bar{1}]$.

Next, consider the new double coset [01].

$N^{(01)} = N^{01} =< (2,3,4)(\bar{2},\bar{3},\bar{4}) >= \{e,(2,3,4)(\bar{2},\bar{3},\bar{4}),(2,4,3)(\bar{2},\bar{4},\bar{3})\}$.

Therefore, the number of single cosets in the double coset [01] are at most

$\frac{|N|}{|N^{(01)}|}=\frac{5!}{3} = \frac{120}{3} = 40$. Each single coset has only one name.

By conjugating Nt_0t_1 with the elements of transversal of $N^{(01)}$, we obtain the following ten single cosets of double coset [01]:
$01, 12, \bar{0}\bar{1}, 23, \bar{1}\bar{2}, 34, \bar{2}\bar{4}, \bar{2}\bar{3}, 40, \bar{4}\bar{3}, \bar{3}\bar{0}, \bar{3}\bar{4}, 24, \bar{0}\bar{4}, \bar{4}\bar{1}, \bar{4}\bar{0}, 43, 30, \bar{1}\bar{0}, 03, \bar{0}\bar{2}, 31, 04, 41, \bar{2}\bar{1}, 10,$
$14, \bar{1}\bar{3}, 02, 42, \bar{0}\bar{3}, \bar{3}\bar{1}, \bar{3}\bar{2}, 21, 20, 13, \bar{1}\bar{4}, \bar{4}\bar{2}, 32, \bar{2}\bar{0}$.

The orbits of $N^{(01)}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{1\},\{0\},\{\bar{1}\},\{\bar{0}\},\{2,3,4\}$ and $\{\bar{2},\bar{3},\bar{4}\}$. Now, take one representative from each orbit, namely 0, $\bar{0}$, 1, $\bar{1}$, 2 and $\bar{2}$, then find the double cosets to which $Nt_0t_1t_1$, $Nt_0t_1\bar{t_1}$, $Nt_0t_1t_0$, $Nt_0t_1\bar{t_0}$, $Nt_0t_1t_2$ and $Nt_0t_1\bar{t_2}$ belong. However,

$Nt_0t_1t_1 = Nt_0\bar{t_1} \in [0\bar{1}]$.
$Nt_0t_1\bar{t_1} = Nt_0 \in [0]$.
$Nt_0t_1t_0 = Nt_0t_1t_0 \in Nt_0t_1t_0N$ (New double coset denote by [010]).
$Nt_0t_1\bar{t_0} = Nt_0t_1\bar{t_0} \in [01]$ **(By Lemma-1)**
$Nt_0t_1t_2 = Nt_0t_1t_2 \in Nt_0t_1t_2N$ (New double coset denote by [012]).
$Nt_0t_1\bar{t_2} = Nt_0t_1\bar{t_2} \in Nt_0t_1\bar{t_2}N$ (New double coset denote by [01$\bar{2}$]).

Consider the double coset [0$\bar{1}$]
Claim: $1\bar{0} \sim 0\bar{1}$

Proof. By our relation-I, $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtain the following relation.
$((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0}\}} = (\bar{0}1)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}}$
$\Rightarrow (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0} = 0\bar{1}$ □

Since $N^{(0\bar{1})} \geq N^{0\bar{1}} =< (2,3,4)(\bar{2},\bar{3},\bar{4}) >$ and
$N(t_1\bar{t_0})^{\{(1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3})\}} = N\bar{t_0}t_1 \Rightarrow (1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3}) \in N^{(0\bar{1})}$
$\Rightarrow N^{(0\bar{1})} =< (2,3,4)(\bar{2},\bar{3},\bar{4}),(1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3}) >$

$= \{e, (1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3}), (2,3,4)(\bar{2},\bar{3},\bar{4}), (1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4}),$
$(2,4,3)(\bar{2},\bar{4},\bar{3}), (1,0)(2,4)(\bar{1},\bar{0})(\bar{2},\bar{4})\}$.

Now, take $Nt_0\bar{t_1}$ conjugates with elements of $N^{(0\bar{1})}$, we obtain the following relations:

$$0\bar{1} \sim 1\bar{0}$$

Therefore, the number of single cosets in the double coset $[0\bar{1}]$ are at most

$$\frac{|N|}{|N^{(0\bar{1})}|} = \frac{5!}{6} = \frac{120}{6} = 20.$$ Each single coset has two different names.

Again, by conjugating $0\bar{1} \sim 1\bar{0}$ to elements of the transversal of $N^{(0\bar{1})}$, we obtain the following twenty single cosets:

$0\bar{1} \sim 1\bar{0}, 1\bar{2} \sim 2\bar{1}, \bar{0}1 \sim \bar{1}0, 2\bar{3} \sim 3\bar{2}, \bar{1}2 \sim \bar{2}1, 3\bar{4} \sim 4\bar{3}, \bar{2}4 \sim \bar{4}2, \bar{2}3 \sim \bar{3}2, 4\bar{0} \sim 0\bar{4}, \bar{3}4 \sim \bar{4}3, \bar{3}0 \sim \bar{0}3, 4\bar{2} \sim 2\bar{4}, \bar{4}0 \sim \bar{0}4, \bar{1}4 \sim \bar{4}1, 0\bar{3} \sim 3\bar{0}, \bar{2}0 \sim \bar{0}2, 1\bar{3} \sim 3\bar{1}, 1\bar{4} \sim 4\bar{1}, \bar{1}3 \sim \bar{3}1, 2\bar{0} \sim 0\bar{2}$.

The orbits of $N^{(0\bar{1})}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{1,0\}, \{\bar{1},\bar{0}\}, \{2,3,4\}$ and $\{\bar{2},\bar{3},\bar{4}\}$. Now, take one representative from each orbit, namely 1, $\bar{1}$, 2 and $\bar{2}$, then find the double cosets to which $Nt_0\bar{t_1}t_1$, $Nt_0\bar{t_1}\bar{t_1}$, $Nt_0\bar{t_1}t_2$ and $Nt_0\bar{t_1}\bar{t_2}$ belong. However,

$Nt_0\bar{t_1}t_1 = Nt_0 \in [0]$
$Nt_0\bar{t_1}\bar{t_1} = Nt_0t_1 \in [01]$
$Nt_0\bar{t_1}t_2 = Nt_0\bar{t_1}t_2 \in Nt_0\bar{t_1}t_2N$ (New double coset denote by $[0\bar{1}2]$.
$Nt_0\bar{t_1}\bar{t_2} = Nt_0\bar{t_1}\bar{t_2} \in Nt_0\bar{t_1}\bar{t_2}N$ (New double coset denote by $[0\bar{1}\bar{2}]$.

Consider the double coset [010].
Claim: $010 \sim \bar{0}\bar{1}\bar{0} \sim 101$

Proof. By our relation-I, $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtain the following relation:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}} = (\bar{0}1)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}}$
  $\Rightarrow (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0} = 0\bar{1}$

From the relation above, we have
1. $010 = \bar{0}\underline{0}\underline{1}0 = \bar{0}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}00 = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}\bar{0}$.

2. $010 = 0\bar{1}\underline{\bar{1}0} = 0\bar{1}(2,4,3)(\bar{2},\bar{4},\bar{3})\bar{0}1 = (2,4,3)(\bar{2},\bar{4},\bar{3})\underline{0\bar{1}}\bar{0}1$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})(2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}\bar{0}1 = (2,3,4)(\bar{2},\bar{3},\bar{4})101.$ □

Since, $N^{(010)} \geq N^{010} = <(2,3,4)(\bar{2},\bar{3},\bar{4})>$ and
$N(t_0t_1t_0)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}} = \bar{0}\bar{1}\bar{0} \Rightarrow (1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0}) \in N^{(010)}$.
$N(t_0t_1t_0)^{\{(1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3})\}} = 010 \Rightarrow (1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3}) \in N^{(010)}$.
$\Rightarrow N^{(010)} = <(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0}), (1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3})>$
$= \{e, (1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3}), (1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0}), (1,\bar{0})(2,\bar{4},3,\bar{2},4,\bar{3})(0,\bar{1}),$
$(2,4,3)(\bar{2},\bar{4},\bar{3}), (1,0)(2,4)(\bar{1},\bar{0})(\bar{2},\bar{4}), (1,\bar{1})(2,\bar{3})(3,\bar{2})(4,\bar{4})(0,\bar{0}),$
$(1,\bar{0})(2,\bar{3},4,\bar{2},3,\bar{4})(0,\bar{1}), (2,3,4)(\bar{2},\bar{3},\bar{4}), (1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4}),$
$(1,\bar{1})(2,\bar{4})(3,\bar{3})(4,\bar{2})(0,\bar{0}), (1,\bar{0})(2,\bar{2})(3,\bar{3})(4,\bar{4})(0,\bar{1})\}$
Take $Nt_0t_1t_0$ conjugates with elements of $N^{(010)}$, we obtain the following relations:

$$010 \sim 101 \sim \bar{0}\bar{1}\bar{0} \sim \bar{1}\bar{0}\bar{1}$$

Therefore, the number of single cosets in the double coset [010] are at most

$\frac{|N|}{|N^{(010)}|} = \frac{5!}{12} = \frac{120}{12} = 10$. Each single coset has four different names.

By conjugating $010 \sim 101 \sim \bar{0}\bar{1}\bar{0} \sim \bar{1}\bar{0}\bar{1}$ with the elements of transversal of $N^{(010)}$, we obtain the following ten single cosets of double coset [010]:

$$\begin{gathered}
010 \sim 101 \sim \bar{0}\bar{1}\bar{0} \sim \bar{1}\bar{0}\bar{1} \\
121 \sim 212 \sim \bar{1}\bar{2}\bar{1} \sim \bar{2}\bar{1}\bar{2} \\
232 \sim 323 \sim \bar{2}\bar{3}\bar{2} \sim \bar{3}\bar{2}\bar{3} \\
343 \sim 434 \sim \bar{3}\bar{4}\bar{3} \sim \bar{4}\bar{3}\bar{4} \\
\bar{2}\bar{4}\bar{2} \sim \bar{4}\bar{2}\bar{4} \sim 242 \sim 424 \\
404 \sim 040 \sim \bar{4}\bar{0}\bar{4} \sim \bar{0}\bar{4}\bar{0} \\
\bar{3}\bar{0}\bar{3} \sim \bar{0}\bar{3}\bar{0} \sim 303 \sim 030 \\
\bar{4}\bar{1}\bar{4} \sim \bar{1}\bar{4}\bar{1} \sim 414 \sim 141 \\
\bar{0}\bar{2}\bar{0} \sim \bar{2}\bar{0}\bar{2} \sim 020 \sim 202 \\
\bar{1}\bar{3}\bar{1} \sim \bar{3}\bar{1}\bar{3} \sim 131 \sim 313
\end{gathered}$$

The orbits of $N^{(010)}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{0,1,\bar{1},\bar{0}\}$ and $\{2,3,4,\bar{2},\bar{3},\bar{4}\}$. Now, take one representative from each orbit, namely 2 and $\bar{0}$, then find the double cosets to which $Nt_0t_1t_0t_{\bar{0}}$ and $Nt_0t_1t_0t_2$ belong. However,

$Nt_0t_1t_0\bar{t_0} = Nt_0t_1 \in [01]$.

$Nt_0t_1t_0t_2 = Nt_0t_1t_0t_2 \in Nt_0t_1t_0t_2N$ (New double coset denote by [0102]).

Next, consider the double coset [012].

$N^{(012)} = N^{012} =< e >$.

Therefore, the number of single cosets in the double coset [012] are at most

$\frac{|N|}{|N^{(012)}|}=\frac{5!}{1} = \frac{120}{1} = 120$. Each single coset has only single name.

$\Rightarrow T \cong S_5$

By conjugating 012 with the elements of transversal of $N^{(012)}$, we obtain the 120 single cosets of double coset [012]:

$012, 312, 412, 410, 310, 210, 214, 312, 013, 012, 413, 213, 123, 423, 023, 021, 421, 321, 320,$
$420, 120, 124, 024, 324, 401, 201, 301, 304, 204, 104, 103, 203, 403, 402, 302, 102, 234, 034,$
$134, 132, 032, 432, 431, 031, 231, 230, 130, 430, 340, 140, 240, 243, 143, 043, 042, 142, 342,$
$341, 241, 041, \bar{0}\bar{1}\bar{2}, \bar{4}\bar{1}\bar{2}, \bar{3}\bar{1}\bar{2}, \bar{3}\bar{1}\bar{0}, \bar{4}\bar{1}\bar{0}, \bar{2}\bar{1}\bar{0}, \bar{2}\bar{1}\bar{3}, \bar{4}\bar{1}\bar{3}, \bar{0}\bar{1}\bar{3}, \bar{0}\bar{1}\bar{4}, \bar{3}\bar{1}\bar{4}, \bar{2}\bar{1}\bar{4}, \bar{1}\bar{2}\bar{4}, \bar{3}\bar{2}\bar{4}, \bar{0}\bar{2}\bar{4}, \bar{0}\bar{2}\bar{1},$
$\bar{4}\bar{2}\bar{1}, \bar{3}\bar{2}\bar{1}, \bar{4}\bar{2}\bar{0}, \bar{3}\bar{2}\bar{0}, \bar{1}\bar{2}\bar{0}, \bar{1}\bar{2}\bar{3}, \bar{0}\bar{2}\bar{3}, \bar{4}\bar{2}\bar{3}, \bar{4}\bar{0}\bar{1}, \bar{3}\bar{0}\bar{1}, \bar{2}\bar{0}\bar{1}, \bar{2}\bar{0}\bar{4}, \bar{3}\bar{0}\bar{4}, \bar{1}\bar{0}\bar{4}, \bar{1}\bar{0}\bar{2}, \bar{3}\bar{0}\bar{2}, \bar{4}\bar{0}\bar{2}, \bar{4}\bar{0}\bar{3}, \bar{2}\bar{0}\bar{3},$
$\bar{1}\bar{0}\bar{3}, \bar{2}\bar{4}\bar{3}, \bar{0}\bar{4}\bar{3}, \bar{1}\bar{4}\bar{3}, \bar{1}\bar{4}\bar{2}, \bar{0}\bar{4}\bar{2}, \bar{3}\bar{4}\bar{2}, \bar{3}\bar{4}\bar{1}, \bar{0}\bar{4}\bar{1}, \bar{2}\bar{4}\bar{1}, \bar{2}\bar{4}\bar{0}, \bar{1}\bar{4}\bar{0}, \bar{3}\bar{4}\bar{0}, \bar{2}\bar{3}\bar{0}, \bar{4}\bar{3}\bar{0}, \bar{1}\bar{3}\bar{0}, \bar{1}\bar{3}\bar{2}, \bar{4}\bar{3}\bar{2}, \bar{0}\bar{3}\bar{2},$
$\bar{0}\bar{3}\bar{1}, \bar{4}\bar{3}\bar{1}, \bar{2}\bar{3}\bar{1}, \bar{2}\bar{3}\bar{4}, \bar{1}\bar{3}\bar{4}, \bar{0}\bar{3}\bar{4}.$

The orbits of $N^{(012)}$ on $\{0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$ and $\{\bar{4}\}$. Now, take one representative from each orbit, namely 0, 1, 2, 3, 4, $\bar{0}$, $\bar{1}$,$\bar{2}$,$\bar{3}$ and $\bar{4}$, then find the double cosets to which $Nt_0t_1t_2t_0$, $Nt_0t_1t_2t_1$, $Nt_0t_1t_2t_2$, $Nt_0t_1t_2t_3$, $Nt_0t_1t_2t_4$, $Nt_0t_1t_2\bar{t_0}$, $Nt_0t_1t_2\bar{t_1}$, $Nt_0t_1t_2\bar{t_2}$, $Nt_0t_1t_2\bar{t_3}$, $Nt_0t_1t_2\bar{t_4}$ belong. However,

$Nt_0t_1t_2t_0 = Nt_0t_1t_2t_0 \in Nt_0t_1t_2t_0N$ (New double coset denote by [0120]).

$Nt_0t_1t_2t_1 = Nt_0t_1t_2t_1 \in Nt_0t_1t_2t_1N$ (New double coset denote by [0121]).

$Nt_0t_1t_2t_2 = Nt_0t_1\bar{t_2} \in [01\bar{2}]$.

$Nt_0t_1t_2t_3 = Nt_0t_1t_2t_3 \in Nt_0t_1t_2t_3N$ (New double coset denote by [0123]).

$Nt_0t_1t_2t_4 = Nt_0t_1t_2t_4 \in [0123]$ **(By Lemma-2)**.

$Nt_0t_1t_2\bar{t_0} = Nt_0t_1t_2\bar{t_0} \in [0102]$ **(By Lemma-3)**.

$Nt_0t_1t_2\bar{t_1} = Nt_0t_1t_2\bar{t_1} \in [0\bar{1}\bar{2}]$ **(By Lemma-4)**.

$Nt_0t_1t_2\bar{t_2} = Nt_0t_1 \in [01]$.

$Nt_0t_1t_2\bar{t_3} = Nt_0t_1t_2\bar{t_3} \in Nt_0t_1t_2\bar{t_3}N$ (New double coset denote by $[012\bar{3}]$).

$Nt_0t_1t_2\bar{t_4} = Nt_0t_1t_2\bar{t_4} \in [012\bar{3}]$ **(By Lemma-5)**.

Consider the double coset $01\bar{2}$

Claim: $01\bar{2} \sim 42\bar{1}$

Proof. By our relation-I, $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtain the following relation:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{2},4,\bar{0})(2,\bar{4},0,\bar{1})(3,\bar{3})\}} = (\bar{0}1)^{\{(1,\bar{7},4,\bar{0})(2,\bar{4},0,\bar{1})(3,\bar{3})\}}$
  $\Rightarrow (0,4,3)(\bar{0},\bar{4},\bar{3})2\bar{1} = 1\bar{2}$

From the relation above, we have
$01\bar{2} = 0(0,4,3)(\bar{0},\bar{4},\bar{3})2\bar{1} = (0,4,3)(\bar{0},\bar{4},\bar{3})42\bar{1}.$ □

Since, $N^{(01\bar{2})} \geq N^{01\bar{2}} =< e >$, and
$N(t_0Nt_1\bar{t_2})^{\{(1,2)(4,0)(\bar{1},\bar{2})(\bar{4},\bar{0})\}} = 42\bar{1} \Rightarrow (1,2)(4,0)(\bar{1},\bar{2})(\bar{4},\bar{0}) \in N^{(01\bar{2})}$
$\Rightarrow N^{(01\bar{2})} =< (1,2)(4,0)(\bar{1},\bar{2})(\bar{4},\bar{0}) >= \{e, (1,2)(4,0)(\bar{1},\bar{2})(\bar{4},\bar{0})\}.$

Therefore, the number of single cosets in the double coset $[01\bar{2}]$ are at most

$\frac{|N|}{|N^{(01\bar{2})}|} = \frac{5!}{2} = \frac{120}{2} = 60$. Each single coset has two different names.

By conjugating ($01\bar{2} \sim 42\bar{1}$) with the elements of transversal of $N^{(01\bar{2})}$, we obtain the following sixty single cosets of double coset $[01\bar{2}]$:
$01\bar{2} \sim 42\bar{1}, 12\bar{3} \sim 03\bar{2}, \bar{0}\bar{1}2 \sim \bar{3}\bar{2}1, 23\bar{4} \sim 14\bar{3}, \bar{1}\bar{2}4 \sim \bar{0}\bar{4}2, \bar{1}\bar{2}3 \sim \bar{4}\bar{3}2, 34\bar{0} \sim 20\bar{4}, \bar{2}\bar{4}3 \sim \bar{1}\bar{3}4, \bar{2}\bar{3}0 \sim \bar{1}\bar{0}3, 12\bar{4} \sim 34\bar{2}, 40\bar{1} \sim 31\bar{0}, \bar{4}\bar{3}0 \sim \bar{2}\bar{0}3, \bar{2}\bar{4}0 \sim \bar{3}\bar{0}4, \bar{3}\bar{4}1 \sim \bar{2}\bar{1}4, 24\bar{0} \sim \bar{0}\bar{4}, \bar{3}\bar{4}0 \sim \bar{1}\bar{0}4, 24\bar{3} \sim 03\bar{4}, 23\bar{0} \sim 40\bar{3}, \bar{3}\bar{0}1 \sim \bar{4}\bar{1}0, \bar{0}\bar{4}1 \sim \bar{3}\bar{1}4, \bar{4}\bar{0}2 \sim \bar{3}\bar{2}0, 43\bar{1} \sim 21\bar{3}, 30\bar{1} \sim 21\bar{0}, \bar{4}\bar{0}1 \sim \bar{2}\bar{1}0, 43\bar{0} \sim \bar{0}\bar{3}, 30\bar{4} \sim 14\bar{0}, 01\bar{4} \sim 34\bar{1}, \bar{4}\bar{1}2 \sim \bar{0}\bar{2}1, \bar{1}\bar{0}2 \sim \bar{4}\bar{2}0, 03\bar{1} \sim 41\bar{3}, \bar{0}\bar{1}3 \sim \bar{4}\bar{3}1, 30\bar{2} \sim 42\bar{0}, 32\bar{4} \sim 04\bar{2}, 41\bar{2} \sim 32\bar{1}, 04\bar{1} \sim 21\bar{4}, 20\bar{1} \sim 41\bar{0}, \bar{4}\bar{0}3 \sim \bar{1}\bar{3}0, 12\bar{0} \sim 40\bar{2}, \bar{1}\bar{3}2 \sim \bar{0}\bar{2}3, 31\bar{2} \sim 02\bar{1}, \bar{2}\bar{1}3 \sim \bar{0}\bar{3}1, 32\bar{0} \sim \bar{0}\bar{2}, 14\bar{2} \sim 02\bar{4}, \bar{2}\bar{1}3 \sim \bar{0}\bar{3}1, 32\bar{0} \sim \bar{0}\bar{2}, 14\bar{2} \sim 02\bar{4}, \bar{4}\bar{2}3 \sim \bar{0}\bar{3}2, 02\bar{3} \sim 43\bar{2}, \bar{3}\bar{1}2 \sim \bar{4}\bar{2}1, \bar{2}\bar{0}1 \sim \bar{3}\bar{1}0, \bar{0}\bar{1}4 \sim \bar{2}\bar{4}1, 01\bar{3} \sim 23\bar{1}, \bar{1}\bar{2}0 \sim \bar{3}\bar{0}2, 13\bar{2} \sim 42\bar{3}, \bar{3}\bar{2}4 \sim \bar{1}\bar{4}2, 20\bar{3} \sim 13\bar{0}, \bar{1}\bar{4}3 \sim \bar{0}\bar{3}4, 13\bar{4} \sim 04\bar{3}, \bar{0}\bar{2}4 \sim \bar{3}\bar{4}2, \bar{2}\bar{3}1 \sim \bar{4}\bar{1}3, 24\bar{1} \sim 31\bar{4}, \bar{2}\bar{0}4 \sim \bar{1}\bar{4}0.$

The orbits of $N^{(01\bar{2})}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{3\}$, $\{\bar{3}\}$, $\{1,2\}$, $\{4,0\}$, $\{\bar{1},\bar{2}\}$ and $\{\bar{4},\bar{0}\}$. Now, take one representative from each orbit, namely 3, $\bar{3}$, 2, 0, $\bar{2}$ and $\bar{0}$ then find the double coset to which $Nt_0t_1\bar{t_2}t_3$, $Nt_0t_1\bar{t_2}\bar{t_3}$, $Nt_0t_1\bar{t_2}t_2$, $Nt_0t_1\bar{t_2}t_0$, $Nt_0t_1\bar{t_2}\bar{t_2}$, $Nt_0t_1\bar{t_2}\bar{t_0}$

belong. However,

$Nt_0t_1\bar{t_2}t_3 = Nt_0t_1\bar{t_2}t_3 \in [01\bar{2}]$ **(By Lemma-6).**
$Nt_0t_1\bar{t_2}\bar{t_3} = Nt_0t_1\bar{t_2}\bar{t_3} \in Nt_0t_1\bar{t_2}\bar{t_3}N$ (New double coset denote by $[01\bar{2}\bar{3}]$).
$Nt_0t_1\bar{t_2}t_2 = Nt_0t_1 \in [01]$.
$Nt_0t_1\bar{t_2}t_0 = Nt_0t_1\bar{t_2}t_0 \in [01\bar{2}]$ **(By Lemma-7).**
$Nt_0t_1\bar{t_2}\bar{t_2} = Nt_0t_1t_2 \in [012]$.
$Nt_0t_1\bar{t_2}\bar{t_0} = Nt_0t_1\bar{t_2}\bar{t_0} \in Nt_0t_1\bar{t_2}\bar{t_0}N$ (New double coset denote by $[01\bar{2}\bar{0}]$).

Consider the double coset $[0\bar{1}2]$
Claim: $0\bar{1}2 \sim 3\bar{2}1 \sim 4\bar{2}0$

Proof. By our relation-I, $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtain the following relation:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,2,3,4,0)(\bar{1},\bar{2},\bar{3},\bar{4},\bar{0},)\}} = (\bar{0}1)^{\{(1,2,3,4,0)(\bar{1},\bar{2},\bar{3},\bar{4},\bar{0},)\}}$
$\Rightarrow (3,4,0)(\bar{3},\bar{4},\bar{0})\bar{2}1 = \bar{1}2$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{2})(2,\bar{1})(3,\bar{4},0,\bar{3},4,\bar{0})\}} = (\bar{0}1)^{\{(1,\bar{2})(2,\bar{1})(3,\bar{4},0,\bar{3},4,\bar{0})\}}$
$\Rightarrow (1,4,0)(\bar{1},\bar{4},\bar{0})2\bar{3} = 3\bar{2}$

From the relations above, we have
1. $0\underline{\bar{1}2} = 0(3,4,0)(\bar{3},\bar{4},\bar{0})\bar{2}1 = (3,4,0)(\bar{3},\bar{4},\bar{0})3\bar{2}1$.
2. $0\underline{\bar{1}2} = 0(3,4,0)(\bar{3},\bar{4},\bar{0})\bar{2}1 = (3,4,0)(\bar{3},\bar{4},\bar{0})\underline{3\bar{2}}1 = (3,4,0)(\bar{3},\bar{4},\bar{0})(1,4,0)(\bar{1},\bar{4},\bar{0})2\bar{3}1$
$= (1,4)(3,0)(\bar{1},\bar{4})(\bar{3},\bar{0})2\bar{3}1$. □

Since, $N^{(0\bar{1}2)} \geq N^{0\bar{1}2} =< e >$, and
$N(t_0\bar{t_1}t_2)^{\{(1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0})\}} = 3\bar{2}1 \Rightarrow (1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0}) \in N^{(0\bar{1}2)}$.
$N(t_0\bar{t_1}t_2)^{\{(1,3,4,0,2)(\bar{1},\bar{3},\bar{4},\bar{0},\bar{2})\}} = 2\bar{3}1 \Rightarrow (1,3,4,0,2)(\bar{1},\bar{3},\bar{4},\bar{0},\bar{2}) \in N^{(0\bar{1}2)}$.
$\Rightarrow N^{(0\bar{1}2)} =< (1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0}), (1,3,4,0,2)(\bar{1},\bar{3},\bar{4},\bar{0},\bar{2}) >$
$= \{e, (1,2,0,4,3)(\bar{1},\bar{2},\bar{0},\bar{4},\bar{3}), (1,3,4,0,2)(\bar{1},\bar{3},\bar{4},\bar{0},\bar{2}), (1,0,3,2,4)(\bar{1},\bar{0},\bar{3},\bar{2},\bar{4}),$
$(1,4,2,3,0)(\bar{1},\bar{4},\bar{2},\bar{3},\bar{0}), (2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0}), (1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0}),$
$(1,3)(2,4)(\bar{1},\bar{3})(\bar{2},\bar{4}), (1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4}), (1,4)(2,0)(\bar{1},\bar{4})(\bar{2},\bar{0})\}$.
Now, take $Nt_0\bar{t_1}t_2$ conjugates with elements of $N^{(0\bar{1}2)}$, we obtain the following relations:

$$0\bar{1}2 \sim 1\bar{0}2 \sim 4\bar{1}3 \sim 2\bar{2}1 \sim 4\bar{2}0 \sim 2\bar{3}1 \sim 1\bar{4}3 \sim 0\bar{3}4 \sim 2\bar{4}0 \sim 3\bar{0}4$$

Therefore, the number of single cosets in the double coset $[0\bar{1}2]$ are at most

$\frac{|N|}{|N^{(0\bar{1}2)}|}=\frac{5!}{10}=\frac{120}{10}=12$. Each single coset has ten different names.

Again, by conjugate $0\bar{1}2 \sim 1\bar{0}2 \sim 4\bar{1}3 \sim 2\bar{2}1 \sim 4\bar{2}0 \sim 2\bar{3}1 \sim 1\bar{4}3 \sim 0\bar{3}4 \sim 2\bar{4}0 \sim 3\bar{0}4$ with the elements of transversal of $N^{(0\bar{1}2)}$, we obtain the following twelve single cosets of double coset $[0\bar{1}2]$:

1. $0\bar{1}2 \sim 1\bar{0}2 \sim 4\bar{1}3 \sim 2\bar{2}1 \sim 4\bar{2}0 \sim 2\bar{3}1 \sim 1\bar{4}3 \sim 0\bar{3}4 \sim 2\bar{4}0 \sim 3\bar{0}4$
2. $1\bar{2}3 \sim 2\bar{1}3 \sim 0\bar{2}4 \sim 4\bar{3}2 \sim 0\bar{3}1 \sim 3\bar{4}2 \sim 2\bar{0}4 \sim 1\bar{4}0 \sim 3\bar{0}1 \sim 4\bar{1}0$
3. $\bar{2}3\bar{0} \sim \bar{3}2\bar{0} \sim \bar{1}3\bar{4} \sim \bar{4}0\bar{3} \sim \bar{1}0\bar{2} \sim \bar{0}4\bar{3} \sim \bar{3}1\bar{4} \sim \bar{2}4\bar{1} \sim \bar{0}1\bar{2} \sim \bar{4}2\bar{1}$
4. $2\bar{3}4 \sim 3\bar{2}4 \sim 1\bar{3}0 \sim 0\bar{4}3 \sim 1\bar{4}2 \sim 4\bar{0}3 \sim 3\bar{1}0 \sim 2\bar{0}1 \sim 4\bar{1}2 \sim 0\bar{2}1$
5. $\bar{1}2\bar{4} \sim \bar{2}1\bar{4} \sim \bar{0}2\bar{3} \sim \bar{3}4\bar{2} \sim \bar{0}4\bar{1} \sim \bar{4}3\bar{2} \sim \bar{2}0\bar{3} \sim \bar{1}3\bar{0} \sim \bar{4}0\bar{1} \sim \bar{3}1\bar{0}$
6. $\bar{0}3\bar{2} \sim \bar{3}0\bar{2} \sim \bar{4}3\bar{1} \sim \bar{1}2\bar{3} \sim \bar{4}2\bar{0} \sim \bar{2}1\bar{3} \sim \bar{3}4\bar{1} \sim \bar{0}1\bar{4} \sim \bar{2}4\bar{0} \sim \bar{1}0\bar{4}$
7. $3\bar{4}0 \sim 4\bar{3}0 \sim 2\bar{4}1 \sim 1\bar{0}4 \sim 2\bar{0}3 \sim 0\bar{1}4 \sim 4\bar{2}1 \sim 3\bar{1}2 \sim 0\bar{2}3 \sim 1\bar{3}2$
8. $\bar{2}4\bar{3} \sim \bar{4}2\bar{3} \sim \bar{1}4\bar{0} \sim \bar{0}3\bar{4} \sim \bar{1}3\bar{2} \sim \bar{3}0\bar{4} \sim \bar{4}1\bar{0} \sim \bar{2}0\bar{1} \sim \bar{3}1\bar{2} \sim \bar{0}2\bar{1}$
9. $\bar{3}0\bar{1} \sim \bar{0}3\bar{1} \sim \bar{4}0\bar{2} \sim \bar{2}1\bar{0} \sim \bar{4}1\bar{3} \sim \bar{3}2\bar{0} \sim \bar{0}4\bar{2} \sim \bar{3}2\bar{4} \sim \bar{1}4\bar{3} \sim \bar{2}3\bar{4}$
10. $0\bar{4}2 \sim 4\bar{0}2 \sim 3\bar{4}1 \sim 1\bar{2}4 \sim 3\bar{2}0 \sim 2\bar{1}4 \sim 4\bar{3}1 \sim 0\bar{1}3 \sim 2\bar{3}0 \sim 1\bar{0}3$
11. $4\bar{0}1 \sim 0\bar{4}1 \sim 3\bar{0}2 \sim 2\bar{1}0 \sim 3\bar{1}4 \sim 1\bar{2}0 \sim 0\bar{3}2 \sim 4\bar{2}3 \sim 1\bar{3}3 \sim 2\bar{4}3$
12. $\bar{4}3\bar{0} \sim \bar{3}4\bar{0} \sim \bar{2}3\bar{1} \sim \bar{1}0\bar{3} \sim \bar{2}0\bar{4} \sim \bar{0}1\bar{3} \sim \bar{3}2\bar{1} \sim \bar{4}1\bar{2} \sim \bar{0}2\bar{4} \sim \bar{1}4\bar{2}$

The orbits of $N^{(0\bar{1}2)}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{1,2,3,4,0\}$ and $\{\bar{1},\bar{2},\bar{3},\bar{4},\bar{0}\}$. Now, take one representative from each orbit, namely 2 and $\bar{2}$ then find the double cosets to which $Nt_0\bar{t_1}t_2t_2$, $Nt_0\bar{t_1}t_2\bar{t_2}$ belong. However,

$Nt_0\bar{t_1}t_2t_2 = Nt_0\bar{t_1}\bar{t_2} \in [0\bar{1}\bar{2}]$
$Nt_0\bar{t_1}t_2\bar{t_2} = Nt_0\bar{t_1} \in [0\bar{1}]$.

Consider the double coset $[0\bar{1}\bar{2}]$
Claim: $0\bar{1}\bar{2} \sim 1\bar{0}\bar{2}$

Proof. By our relation-I, $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtain the following relation:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}} = (\bar{0}1)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}}$
  $\Rightarrow (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0} = 0\bar{1}$

From the relation above, we have
$\underline{0\bar{1}}\bar{2} = (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}\bar{2}$. □

Since, $N^{(0\bar{1}\bar{2})} \geq N^{0\bar{1}\bar{2}} =< e >$, and
$N(0\bar{1}\bar{2})^{\{(1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4})\}} = 1\bar{0}\bar{2} \Rightarrow (1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4}) \in N^{(0\bar{1}\bar{2})}$.
$\Rightarrow N^{(0\bar{1}\bar{2})} =< (1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4}) >= \{e,(1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4})\}$.
Therefore, the number of single cosets in the double coset $[01\bar{2}]$ are at most

$\frac{|N|}{|N^{(01\bar{2})}|} = \frac{5!}{2} = \frac{120}{2} = 60$. Each single coset has two different names.

Next, by conjugate $0\bar{1}\bar{2} \sim 1\bar{0}\bar{2}$ with the elements of transversal of $N^{(0\bar{1}\bar{2})}$, we obtain the following sixty single cosets of double coset $[0\bar{1}\bar{2}]$:
$0\bar{1}\bar{2} \sim 1\bar{0}\bar{2}, 1\bar{2}\bar{3} \sim 2\bar{1}\bar{3}, \bar{0}12 \sim \bar{1}02, 2\bar{3}\bar{4} \sim 3\bar{2}\bar{4}, \bar{1}24 \sim \bar{2}14, \bar{1}23 \sim \bar{2}13, 3\bar{4}\bar{0} \sim 4\bar{3}\bar{0}, \bar{2}43 \sim \bar{4}23, \bar{2}30 \sim \bar{3}20, \bar{2}34 \sim \bar{3}24, 1\bar{2}\bar{4} \sim 2\bar{1}\bar{4}, 4\bar{0}\bar{1} \sim 0\bar{4}\bar{1}, \bar{3}40 \sim \bar{4}30, \bar{3}04 \sim \bar{0}34, \bar{3}41 \sim \bar{4}31, 2\bar{4}\bar{0} \sim 4\bar{2}\bar{0}, 2\bar{4}\bar{3} \sim 4\bar{2}\bar{3}, 2\bar{3}\bar{0} \sim 3\bar{2}\bar{0}, \bar{3}01 \sim \bar{0}31, \bar{4}01 \sim \bar{0}41, \bar{4}10 \sim \bar{1}40, 4\bar{0}\bar{3} \sim 0\bar{4}\bar{3}, \bar{4}02 \sim \bar{0}42, 4\bar{3}\bar{1} \sim 3\bar{4}\bar{1}, 3\bar{0}\bar{1} \sim 0\bar{3}\bar{1}, 3\bar{0}\bar{4} \sim 0\bar{3}\bar{4}, \bar{2}40 \sim \bar{4}20, \bar{4}12 \sim \bar{1}42, \bar{0}21 \sim \bar{2}01, 3\bar{1}\bar{0} \sim 1\bar{0}\bar{3}, 0\bar{1}\bar{4} \sim 1\bar{0}\bar{4}, \bar{0}13 \sim \bar{1}03, 3\bar{2}\bar{0} \sim 2\bar{3}\bar{0}, 4\bar{0}\bar{2} \sim 0\bar{4}\bar{2}, 4\bar{1}\bar{2} \sim 1\bar{4}\bar{2}, 4\bar{1}\bar{0} \sim 1\bar{4}\bar{0}, \bar{4}03 \sim \bar{0}43, \bar{0}23 \sim \bar{2}03, 3\bar{1}\bar{2} \sim 1\bar{3}\bar{2}, \bar{1}32 \sim \bar{3}12, 2\bar{0}\bar{1} \sim 0\bar{2}\bar{1}, 2\bar{4}\bar{1} \sim 4\bar{2}\bar{1}, 1\bar{2}\bar{0} \sim 2\bar{1}\bar{0}, 1\bar{4}\bar{3} \sim 4\bar{1}\bar{3}, 0\bar{1}\bar{3} \sim 1\bar{0}\bar{3}, \bar{3}02 \sim \bar{0}32, 0\bar{2}\bar{3} \sim 2\bar{0}\bar{3}, \bar{3}10 \sim \bar{1}30, \bar{0}14 \sim \bar{1}04, \bar{1}34 \sim \bar{3}14, 0\bar{2}\bar{4} \sim 2\bar{0}\bar{4}, \bar{2}31 \sim \bar{3}21, 2\bar{3}\bar{1} \sim 3\bar{2}\bar{1}, \bar{1}20 \sim \bar{2}10, \bar{1}43 \sim \bar{4}13, 1\bar{3}\bar{4} \sim 3\bar{1}\bar{4}, \bar{0}24 \sim \bar{2}04, \bar{2}41 \sim \bar{4}21, \bar{4}32 \sim \bar{3}42, 3\bar{4}\bar{2} \sim 4\bar{3}\bar{3}$.

The orbits of $N^{(0\bar{1}\bar{2})}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{2\}$, $\{\bar{2}\}$, $\{1,0\}$, $\{3,4\}$, $\{\bar{1},\bar{0}\}$ and $\{\bar{3},\bar{4}\}$. Now, take one representative from each orbit, namely 2, $\bar{2}$, 0, 3, $\bar{0}$ and $\bar{3}$, then find the double cosets to which $Nt_0\bar{t_1}\bar{t_2}t_2$, $Nt_0\bar{t_1}\bar{t_2}\bar{t_2}$, $Nt_0\bar{t_1}\bar{t_2}t_0$, $Nt_0\bar{t_1}\bar{t_2}t_3$, $Nt_0\bar{t_1}\bar{t_2}\bar{t_0}$ and $Nt_0\bar{t_1}\bar{t_2}\bar{t_3}$ belong. However,

$Nt_0\bar{t_1}\bar{t_2}t_2 = Nt_0\bar{t_1} \in [0\bar{1}]$.
$Nt_0\bar{t_1}\bar{t_2}\bar{t_2} = Nt_0\bar{t_1}t_2 \in [0\bar{1}2]$.
$Nt_0\bar{t_1}\bar{t_2}t_0 = Nt_0\bar{t_1}\bar{t_2}t_0 \in [012]$ **(By Lemma-8).**
$Nt_0\bar{t_1}\bar{t_2}t_3 = Nt_0\bar{t_1}\bar{t_2}t_3 \in Nt_0\bar{t_1}\bar{t_2}t_3N$ (New double coset denote by $[0\bar{1}\bar{2}3]$).
$Nt_0\bar{t_1}\bar{t_2}\bar{t_0} = Nt_0\bar{t_1}\bar{t_2}\bar{t_0} \in [0121]$ **(By Lemma-9).**
$Nt_0\bar{t_1}\bar{t_2}\bar{t_3} = Nt_0\bar{t_1}\bar{t_2}\bar{t_3} \in Nt_0\bar{t_1}\bar{t_2}\bar{t_3}N$ (New double coset denote by $[0\bar{1}\bar{2}\bar{3}]$).

Consider double coset [0102]
Claim: $0102 \sim 1012$

Proof. By our relation-I, $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtain the following relation:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}} = (\bar{0}1)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}}$
  $\Rightarrow (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0} = 0\bar{1}$

From the relation above, we have
$0102 = \underline{0\bar{1}}\bar{1}02 = (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}\underline{\bar{1}0}2 = (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}(2,4,3)(\bar{2},\bar{4},\bar{3})\bar{0}12$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})1012.$ □

Since, $N^{(0102)} \geq N^{0102} =< e >$, and
$N(t_0t_1t_0t_2)^{\{(1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4})\}} = Nt_1t_0t_1t_2 \Rightarrow (1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4}) \in N^{(0102)}$.
$\Rightarrow N^{(0102)} =< (1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4}) >= \{e,(1,0)(3,4)(\bar{1},\bar{0})(\bar{3},\bar{4})\}$.

Therefore, the number of single cosets in the double coset $[01\bar{2}]$ are at most

$\frac{|N|}{|N^{(0102)}|} = \frac{5!}{2} = \frac{120}{2} = 60$. Each single coset has two different names.

Again, by conjugate $0102 \sim 1012$ with the elements of transversal of $N^{(0102)}$, we obtain the sixty single cosets of double coset [0102]:

$0102 \sim 1012, 1212 \sim 2123, 2324 \sim 3234, 3430 \sim 4340, 1214 \sim 2124, 4041 \sim 0401,$
$2420 \sim 4240, 2423 \sim 4243, 2320 \sim 3230, 4043 \sim 0403, 4341 \sim 3431, 3031 \sim 0301,$
$3034 \sim 0304, 3130 \sim 1310, 0104 \sim 1014, 3032 \sim 0302, 4042 \sim 0402, 4142 \sim 1412,$
$4140 \sim 1410, 3132 \sim 1312, 2021 \sim 0201, 2421 \sim 4241, 1210 \sim 2120, 1413 \sim 4143,$
$0103 \sim 1013, 0203 \sim 2023, 0204 \sim 2024, 2321 \sim 3231, 1314 \sim 3134, 3432 \sim 4342,$
$\bar{0}\bar{1}\bar{0}\bar{2} \sim \bar{1}\bar{0}\bar{1}\bar{2}, \bar{1}\bar{2}\bar{1}\bar{4} \sim \bar{2}\bar{1}\bar{2}\bar{4}, \bar{1}\bar{2}\bar{1}\bar{3} \sim \bar{2}\bar{1}\bar{2}\bar{3}, \bar{2}\bar{4}\bar{2}\bar{3} \sim \bar{4}\bar{2}\bar{4}\bar{3}, \bar{2}\bar{3}\bar{2}\bar{0} \sim \bar{3}\bar{2}\bar{3}\bar{0}, \bar{2}\bar{3}\bar{2}\bar{4} \sim \bar{3}\bar{2}\bar{3}\bar{4},$
$\bar{3}\bar{4}\bar{3}\bar{0} \sim \bar{4}\bar{3}\bar{4}\bar{0}, \bar{3}\bar{0}\bar{3}\bar{4} \sim \bar{0}\bar{3}\bar{0}\bar{4}, \bar{3}\bar{4}\bar{3}\bar{1} \sim \bar{4}\bar{3}\bar{4}\bar{1}, \bar{3}\bar{0}\bar{3}\bar{1} \sim \bar{0}\bar{3}\bar{0}\bar{1}, \bar{4}\bar{0}\bar{4}\bar{1} \sim \bar{0}\bar{4}\bar{0}\bar{1}, \bar{4}\bar{1}\bar{4}\bar{0} \sim \bar{1}\bar{4}\bar{1}\bar{0},$
$\bar{4}\bar{0}\bar{4}\bar{2} \sim \bar{0}\bar{4}\bar{0}\bar{2}, \bar{2}\bar{4}\bar{2}\bar{0} \sim \bar{4}\bar{2}\bar{4}\bar{0}, \bar{4}\bar{1}\bar{4}\bar{2} \sim \bar{1}\bar{4}\bar{1}\bar{2}, \bar{0}\bar{2}\bar{0}\bar{1} \sim \bar{2}\bar{0}\bar{2}\bar{1}, \bar{0}\bar{1}\bar{0}\bar{3} \sim \bar{1}\bar{0}\bar{1}\bar{3}, \bar{4}\bar{0}\bar{4}\bar{3} \sim \bar{0}\bar{4}\bar{0}\bar{3},$
$\bar{0}\bar{2}\bar{0}\bar{3} \sim \bar{2}\bar{0}\bar{2}\bar{3}, \bar{1}\bar{3}\bar{1}\bar{2} \sim \bar{3}\bar{1}\bar{3}\bar{2}, \bar{3}\bar{0}\bar{3}\bar{2} \sim \bar{0}\bar{3}\bar{0}\bar{2}, \bar{3}\bar{1}\bar{3}\bar{0} \sim \bar{1}\bar{3}\bar{1}\bar{0}, \bar{0}\bar{1}\bar{0}\bar{4} \sim \bar{1}\bar{0}\bar{1}\bar{4}, \bar{1}\bar{3}\bar{1}\bar{4} \sim \bar{3}\bar{1}\bar{3}\bar{4},$
$\bar{2}\bar{3}\bar{2}\bar{1} \sim \bar{3}\bar{2}\bar{3}\bar{1}, \bar{1}\bar{2}\bar{1}\bar{0} \sim \bar{2}\bar{1}\bar{2}\bar{0}, \bar{1}\bar{4}\bar{1}\bar{3} \sim \bar{4}\bar{1}\bar{4}\bar{3}, \bar{0}\bar{2}\bar{0}\bar{4} \sim \bar{2}\bar{0}\bar{2}\bar{4}, \bar{2}\bar{4}\bar{2}\bar{1} \sim \bar{4}\bar{2}\bar{4}\bar{1}, \bar{4}\bar{3}\bar{4}\bar{2} \sim \bar{3}\bar{4}\bar{3}\bar{2}.$

The orbits of $N^{(0102)}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{2\}$, $\{\bar{2}\}$, $\{1,0\}$, $\{3,4\}$, $\{\bar{1},\bar{0}\}$ and $\{\bar{3},\bar{4}\}$. Now, take one representative from each orbit, namely 2, $\bar{2}$, 0, 3, $\bar{0}$ and $\bar{3}$, then find the double coset to which $Nt_0t_1t_0t_2t_2$, $Nt_0t_1t_0t_2\bar{t_2}$, $Nt_0t_1t_0t_2t_0$,$Nt_0t_1t_0t_2\bar{t_0}$, $Nt_0t_1t_0t_2t_3$ and $Nt_0t_1t_0t_2\bar{t_3}$ belong. However,

$Nt_0t_1t_0t_2t_2 = Nt_0t_1t_0\bar{t_2} \in [0102]$ **(By Lemma-10).**
$Nt_0t_1t_0t_2\bar{t_2} = Nt_0t_1t_0 \in [010]$.
$Nt_0t_1t_0t_2t_0 = Nt_0t_1t_0t_2t_0 \in [0120]$ **(By Lemma-11).**
$Nt_0t_1t_0t_2\bar{t_0} = Nt_0t_1t_0t_2\bar{t_0} \in [012]$ **(By Lemma-12).**
$Nt_0t_1t_0t_2t_3 = Nt_0t_1t_0t_2t_3 \in Nt_0t_1t_0t_2t_3N$ (New double coset denote by [01023]).
$Nt_0t_1t_0t_2\bar{t_3} = Nt_0t_1t_0t_2\bar{t_3} \in [0\bar{1}\bar{2}3]$ **(By Lemma-13).**

Consider double coset [0120]

$N^{(0120)} = N^{0120} =< e >$

Therefore, the number of single cosets in the double coset [0120] are at most

$$\frac{|N|}{|N^{(0120)}|} = \frac{5!}{1} = \frac{120}{1} = 120. \text{ Each single coset has only one name.}$$

$\Rightarrow T \cong S_5$.

Now, conjugating 0120 with elements of transversal of $N^{(0120)}$, we obtain the 120 single cosets of double coset [0120]:

0120, 3132, 4124, 4104, 3103, 2102, 2142, 3143, 0140, 0130, 4134, 2132, 1231, 4234, 0230, 0210, 4214, 3213, 3203, 4204, 1201, 1241, 0240, 3243, 4014, 2012, 3013, 3043, 2042, 1041, 1031, 2032, 4034, 4024, 3023, 1021, 2342, 0340, 1341, 1321, 0320, 4324, 4314, 0310, 2312, 2302, 1301, 4304, 3403, 1401, 2402, 2432, 1431, 0430, 0420, 1421, 3423, 3413, 2412, 0410,
$\bar0\bar1\bar2\bar0, \bar4\bar1\bar2\bar4, \bar3\bar1\bar2\bar3, \bar3\bar1\bar0\bar3, \bar4\bar1\bar0\bar4, \bar2\bar1\bar0\bar2, \bar2\bar1\bar3\bar2, \bar4\bar1\bar3\bar4, \bar0\bar1\bar3\bar0, \bar0\bar1\bar4\bar0, \bar3\bar1\bar4\bar3, \bar2\bar1\bar4\bar2, \bar1\bar2\bar4\bar1, \bar3\bar2\bar4\bar3, \bar0\bar2\bar4\bar0,$
$\bar0\bar2\bar1\bar0, \bar3\bar2\bar1\bar3, \bar4\bar2\bar1\bar4, \bar4\bar2\bar0\bar4, \bar3\bar2\bar0\bar3, \bar1\bar2\bar0\bar1, \bar1\bar2\bar3\bar1, \bar0\bar2\bar3\bar0, \bar4\bar2\bar3\bar4, \bar4\bar0\bar1\bar4, \bar3\bar0\bar1\bar3, \bar2\bar0\bar1\bar2, \bar2\bar0\bar4\bar2, \bar3\bar0\bar4\bar3, \bar1\bar0\bar4\bar1,$
$\bar1\bar0\bar2\bar1, \bar3\bar0\bar2\bar3, \bar4\bar0\bar2\bar4, \bar4\bar0\bar3\bar4, \bar2\bar0\bar3\bar2, \bar1\bar0\bar3\bar1, \bar2\bar4\bar3\bar2, \bar0\bar4\bar3\bar0, \bar1\bar4\bar3\bar1, \bar1\bar4\bar2\bar1, \bar0\bar4\bar2\bar0, \bar3\bar4\bar2\bar3, \bar3\bar4\bar1\bar3, \bar0\bar4\bar1\bar0, \bar2\bar4\bar1\bar2,$
$\bar2\bar4\bar0\bar2, \bar1\bar4\bar0\bar1, \bar3\bar4\bar0\bar3, \bar2\bar3\bar0\bar2, \bar4\bar3\bar0\bar4, \bar1\bar3\bar0\bar1, \bar1\bar3\bar2\bar1, \bar4\bar3\bar2\bar4, \bar0\bar3\bar2\bar0, \bar0\bar3\bar1\bar0, \bar4\bar3\bar1\bar4, \bar2\bar3\bar1\bar2, \bar2\bar3\bar4\bar2, \bar1\bar3\bar4\bar1, \bar0\bar3\bar4\bar0.$

The orbits of $N^{(0120)}$ on $\{0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}$, $\{\bar{3}\}$ and $\{\bar{4}\}$. Now, take one representative from each orbit, and find the double cosets to which $Nt_0t_1t_2t_0t_0$, $Nt_0t_1t_2t_0t_1$, $Nt_0t_1t_2t_0t_2$, $Nt_0t_1t_2t_0t_3$, $Nt_0t_1t_2t_0t_4$, $Nt_0t_1t_2t_0\bar{t_0}$, $Nt_0t_1t_2t_0\bar{t_1}$, $Nt_0t_1t_2t_0\bar{t_2}$, $Nt_0t_1t_2t_0\bar{t_3}$, $Nt_0t_1t_2t_0\bar{t_4}$ belong. However,

$Nt_0t_1t_2t_0t_0 = Nt_0t_1t_2\bar{t_0} \in [0102]$ **(By Lemma-14).**
$Nt_0t_1t_2t_0t_1 = Nt_0t_1t_2t_0t_1 \in Nt_0t_1t_2t_0t_1N$ (New double coset denote by [01201]).
$Nt_0t_1t_2t_0t_2 = Nt_0t_1t_2t_0t_2 \in [0120]$ **(By Lemma-15).**

$Nt_0t_1t_2t_0t_3 = Nt_0t_1t_2t_0t_3 \in Nt_0t_1t_2t_0t_3N$ (New double coset denote by [01203]).
$Nt_0t_1t_2t_0t_4 = Nt_0t_1t_2t_0t_4 \in Nt_0t_1t_2t_0t_4N$ (New double coset denote by [01204]).
$Nt_0t_1t_2t_0\bar{t_0} = Nt_0t_1t_2 \in [012]$.
$Nt_0t_1t_2t_0\bar{t_1} = Nt_0t_1t_2t_0\bar{t_1} \in [01203]$ **(By Lemma-16).**
$Nt_0t_1t_2t_0\bar{t_2} = Nt_0t_1t_2t_0\bar{t_2} \in [01\bar{2}\bar{0}]$ **(By Lemma-17).**
$Nt_0t_1t_2t_0\bar{t_3} = Nt_0t_1t_2t_0\bar{t_3} \in Nt_0t_1t_2t_0\bar{t_3}N$ (New double coset denote by $[0120\bar{3}]$).
$Nt_0t_1t_2t_0\bar{t_4} = Nt_0t_1t_2t_0\bar{t_4} \in [0120\bar{3}]$ **(By Lemma-18).**

Consider the double coset [0121]

Claim: $0121 \sim 3212$

Proof. By our relation-I, $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtain the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,2,3,4,0)(\bar{1},\bar{2},\bar{3},\bar{4},\bar{0})\}} = (\bar{0}1)^{\{(1,2,3,4,0)(\bar{1},\bar{2},\bar{3},\bar{4},\bar{0})\}}$
  $\Rightarrow (3,4,0)(\bar{3},\bar{4},\bar{0})\bar{2}1 = \bar{1}2$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{2},4,\bar{0})(2,\bar{4},0,\bar{1})(3,\bar{3})\}} = (\bar{0}1)^{\{(1,\bar{7},4,\bar{0})(2,\bar{4},0,\bar{1})(3,\bar{3})\}}$
  $\Rightarrow (0,4,3)(\bar{0},\bar{4},\bar{3})2\bar{1} = 1\bar{2}$

From the relations above, we have
$0121 = 01\bar{2}\underline{\bar{2}1} = 01\bar{2}(3,0,4)(\bar{3},\bar{0},\bar{4})\bar{1}2 = (3,0,4)(\bar{3},\bar{0},\bar{4})4\underline{1\bar{2}}\bar{1}2$
$= (3,0,4)(\bar{3},\bar{0},\bar{4})4(0,4,3)(\bar{0},\bar{4},\bar{3})2\bar{1}\bar{1}2 = (3,4,0)(\bar{3},\bar{4},\bar{0})32\bar{1}\bar{1}2 = (3,4,0)(\bar{3},\bar{4},\bar{0})3212.$ □

Since, $N^{(0121)} \geq N^{0121} =< e >$, and
$N(t_0t_1t_2t_1)^{\{(1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0})\}} = Nt_3t_2t_1t_2 \Rightarrow (1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0}) \in N^{(0121)}$.
$\Rightarrow N^{(0121)} =< (1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0}) >= \{e, (1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0})\}$.

Therefore, the number of single cosets in the double coset [0121] are at most

$\frac{|N|}{|N^{(0121)}|} = \frac{5!}{2} = \frac{120}{2} = 60$. Each single coset has two different names.

By conjugating ($0121 \sim 3212$) with the elements of transversal of $N^{(0121)}$, we obtain the following sixty single cosets of double coset [0121]:
$0121 \sim 3212, 1232 \sim 4323, 2343 \sim 0434, 3404 \sim 1040, 1242 \sim 0424, 4010 \sim 2101,$
$2404 \sim 3040, 2434 \sim 1343, 2303 \sim 1030, 4030 \sim 1303, 0131 \sim 4313, 3010 \sim 4010,$

$4303 \sim 2030, 3414 \sim 2141, 0313 \sim 2131, 2010 \sim 3101, 0141 \sim 2414, 1202 \sim 3020,$
$4121 \sim 0212, 0414 \sim 3141, 4020 \sim 3202, 3121 \sim 4212, 1020 \sim 4202, 1424 \sim 3242,$
$2313 \sim 4131, 1323 \sim 0232, 0242 \sim 3424, 4232 \sim 0323, 1434 \sim 0343, 1404 \sim 2040,$
$\overline{0121} \sim \overline{4212}, \overline{1242} \sim \overline{3424}, \overline{1232} \sim \overline{0323}, \overline{2434} \sim \overline{0343}, \overline{2303} \sim \overline{4030}, \overline{2343} \sim \overline{1434},$
$\overline{4303} \sim \overline{1030}, \overline{3040} \sim \overline{1404}, \overline{3414} \sim \overline{0141}, \overline{2404} \sim \overline{3040}, \overline{3010} \sim \overline{2101}, \overline{0414} \sim \overline{2141},$
$\overline{4101} \sim \overline{2010}, \overline{4020} \sim \overline{1202}, \overline{4010} \sim \overline{3101}, \overline{2404} \sim \overline{1040}, \overline{4121} \sim \overline{3212}, \overline{1020} \sim \overline{3202},$
$\overline{0232} \sim \overline{3121}, \overline{0131} \sim \overline{2313}, \overline{2131} \sim \overline{4313}, \overline{0232} \sim \overline{4323}, \overline{1323} \sim \overline{4232}, \overline{0313} \sim \overline{4131},$
$\overline{3020} \sim \overline{4202}, \overline{3242} \sim \overline{0424}, \overline{1343} \sim \overline{0434}, \overline{2414} \sim \overline{3141}, \overline{1424} \sim \overline{0242}, \overline{1303} \sim \overline{2030}.$

The orbits of $N^{(0121)}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{4\},\{\bar{4}\},\{1,2\},\{3,0\},\{\bar{1},\bar{2}\}$ and $\{\bar{3},\bar{0}\}$. Now, take one representative from each orbit and find the double coset, namely $4, \bar{4}, 1, 0$, $\bar{1}$ and $\bar{0}$, then find the double cosets to which $Nt_0t_1t_2t_1t_4$, $Nt_0t_1t_2t_1\bar{t_4}$, $Nt_0t_1t_2t_1t_1$, $Nt_0t_1t_2t_1t_0$, $Nt_0t_1t_2t_1\bar{t_1}$ and $Nt_0t_1t_2t_1\bar{t_0}$ belong. However,

$Nt_0t_1t_2t_1t_4 = Nt_0t_1t_2t_1t_4 \in Nt_0t_1t_2t_1t_4N$ (New double coset denote by [01214]).
$Nt_0t_1t_2t_1\bar{t_4} = Nt_0t_1t_2t_1\bar{t_4} \in [01204]$ **(By Lemma-19).**
$Nt_0t_1t_2t_1t_1 = Nt_0t_1t_2\bar{t_1} \in [0\bar{1}\bar{2}]$ **(By Lemma-20).**
$Nt_0t_1t_2t_1t_0 = Nt_0t_1t_2t_1t_0 \in [01214]$ **(By Lemma-21).**
$Nt_0t_1t_2t_1\bar{t_1} = Nt_0t_1t_2 \in [012]$.
$Nt_0t_1t_2t_1\bar{t_0} = Nt_0t_1t_2t_1\bar{t_0} \in [01203]$ **(By Lemma-22).**

Consider the double coset [0123]

$N^{(0123)} = N^{0123} = < e >$
Therefore, the number of single cosets in the double coset [0123] are at most

$$\frac{|N|}{|N^{(0123)}|} = \frac{5!}{1} = \frac{120}{1} = 120. \text{ Each single coset has only one name.}$$

$\Rightarrow T \cong S_5$.
Now, conjugating (0123) with elements of transversal of $N^{(0123)}$, we obtain the 120 single cosets of double coset [0123]:
$0123, 3124, 4120, 4103, 3102, 2104, 2143, 3140, 0142, 0134, 4132, 2130, 1234, 4230, 0231,$
$0214, 4213, 3210, 3204, 4201, 1203, 1240, 0243, 3241, 4012, 2013, 3014, 3042, 2041, 1043,$
$1032, 2034, 4031, 4023, 3021, 1024, 2340, 0341, 1342, 1320, 0324, 4321, 4310, 0312, 2314,$

$2301, 1304, 4302, 3401, 1402, 2403, 2431, 1430, 0432, 0421, 1423, 3420, 3412, 2410, 0413,$
$\bar{0}\bar{1}\bar{2}\bar{4}, \bar{4}\bar{1}\bar{2}\bar{3}, \bar{3}\bar{1}\bar{2}\bar{0}, \bar{3}\bar{1}\bar{0}\bar{4}, \bar{4}\bar{1}\bar{0}\bar{2}, \bar{2}\bar{1}\bar{0}\bar{3}, \bar{2}\bar{1}\bar{3}\bar{4}, \bar{4}\bar{1}\bar{3}\bar{0}, \bar{0}\bar{1}\bar{3}\bar{2}, \bar{0}\bar{1}\bar{4}\bar{3}, \bar{3}\bar{1}\bar{4}\bar{2}, \bar{2}\bar{1}\bar{4}\bar{0}, \bar{1}\bar{2}\bar{4}\bar{3}, \bar{3}\bar{2}\bar{4}\bar{0}, \bar{0}\bar{2}\bar{4}\bar{1},$
$\bar{0}\bar{2}\bar{1}\bar{3}, \bar{3}\bar{2}\bar{1}\bar{4}, \bar{4}\bar{2}\bar{1}\bar{0}, \bar{4}\bar{2}\bar{0}\bar{3}, \bar{3}\bar{2}\bar{0}\bar{1}, \bar{1}\bar{2}\bar{0}\bar{4}, \bar{1}\bar{2}\bar{3}\bar{0}, \bar{0}\bar{2}\bar{3}\bar{4}, \bar{4}\bar{2}\bar{3}\bar{1}, \bar{4}\bar{0}\bar{1}\bar{3}, \bar{3}\bar{0}\bar{1}\bar{2}, \bar{2}\bar{0}\bar{1}\bar{4}, \bar{2}\bar{0}\bar{4}\bar{3}, \bar{3}\bar{0}\bar{4}\bar{1}, \bar{1}\bar{0}\bar{4}\bar{2},$
$\bar{1}\bar{0}\bar{2}\bar{3}, \bar{3}\bar{0}\bar{2}\bar{4}, \bar{4}\bar{0}\bar{2}\bar{1}, \bar{4}\bar{0}\bar{3}\bar{2}, \bar{2}\bar{0}\bar{3}\bar{1}, \bar{1}\bar{0}\bar{3}\bar{4}, \bar{2}\bar{4}\bar{3}\bar{0}, \bar{0}\bar{4}\bar{3}\bar{1}, \bar{1}\bar{4}\bar{3}\bar{2}, \bar{1}\bar{4}\bar{2}\bar{0}, \bar{0}\bar{4}\bar{2}\bar{3}, \bar{3}\bar{4}\bar{2}\bar{1}, \bar{3}\bar{4}\bar{1}\bar{0}, \bar{0}\bar{4}\bar{1}\bar{2}, \bar{2}\bar{4}\bar{1}\bar{3},$
$\bar{2}\bar{4}\bar{0}\bar{1}, \bar{1}\bar{4}\bar{0}\bar{3}, \bar{3}\bar{4}\bar{0}\bar{2}, \bar{2}\bar{3}\bar{0}\bar{4}, \bar{4}\bar{3}\bar{0}\bar{1}, \bar{1}\bar{3}\bar{0}\bar{2}, \bar{1}\bar{3}\bar{2}\bar{4}, \bar{4}\bar{3}\bar{2}\bar{0}, \bar{0}\bar{3}\bar{2}\bar{1}, \bar{0}\bar{3}\bar{1}\bar{4}, \bar{4}\bar{3}\bar{1}\bar{2}, \bar{2}\bar{3}\bar{1}\bar{0}, \bar{2}\bar{3}\bar{4}\bar{1}, \bar{1}\bar{3}\bar{4}\bar{0}, \bar{0}\bar{3}\bar{4}\bar{2}.$

The orbits of $N^{(0123)}$ on $\{0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ are $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}$, $\{\bar{3}\}$ and $\{\bar{4}\}$. Now, take one representative from each orbit and find the double coset to which $Nt_0t_1t_2t_3t_0$, $Nt_0t_1t_2t_3t_1$, $Nt_0t_1t_2t_3t_2$, $Nt_0t_1t_2t_3t_3$, $Nt_0t_1t_2t_3t_4$, $Nt_0t_1t_2t_3\bar{t_0}$, $Nt_0t_1t_2t_3\bar{t_1}$, $Nt_0t_1t_2t_3\bar{t_2}$, $Nt_0t_1t_2t_3\bar{t_3}$, $Nt_0t_1t_2t_3\bar{t_4}$ belong. However,

$Nt_0t_1t_2t_3t_0 = Nt_0t_1t_2t_3t_0 \in [01\bar{2}\bar{0}]$ **(By Lemma-23).**
$Nt_0t_1t_2t_3t_1 = Nt_0t_1t_2t_3t_1 \in [01204]$ **(By Lemma-24).**
$Nt_0t_1t_2t_3t_2 = Nt_0t_1t_2t_3t_2 \in [0120\bar{3}]$ **(By Lemma-25).**
$Nt_0t_1t_2t_3t_3 = Nt_0t_1t_2\bar{t_3} \in [012\bar{3}]$.
$Nt_0t_1t_2t_3t_4 = Nt_0t_1t_2t_3t_4 \in [012]$ **(By Lemma-26).**
$Nt_0t_1t_2t_3\bar{t_0} = Nt_0t_1t_2t_3\bar{t_0} \in [0120\bar{3}]$ **(By Lemma-27).**
$Nt_0t_1t_2t_3\bar{t_1} = Nt_0t_1t_2t_3\bar{t_1} \in [01203]$ **(By Lemma-28).**
$Nt_0t_1t_2t_3\bar{t_2} = Nt_0t_1t_2t_3\bar{t_2} \in [01\bar{2}\bar{3}]$ **(By Lemma-29).**
$Nt_0t_1t_2t_3\bar{t_3} = Nt_0t_1t_2 \in [012]$.
$Nt_0t_1t_2t_3\bar{t_4} = Nt_0t_1t_2t_3\bar{t_4} \in [012\bar{3}]$ **(By Lemma-30).**

Consider the double coset $[012\bar{3}]$
Claim: $012\bar{3} \sim 034\bar{1}$

Proof. By our relation-I, $(2, 3, 4)(\bar{2}, \bar{3}, \bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtain the following relations:

- $((2, 3, 4)(\bar{2}, \bar{3}, \bar{4})\bar{1}0)^{\{(1,2,3,4,0)(\bar{1},\bar{2},\bar{3},\bar{4},\bar{0})\}} = (\bar{0}1)^{\{(1,2,3,4,0)(\bar{1},\bar{2},\bar{3},\bar{4},\bar{0})\}}$
  $\Rightarrow (3, 4, 0)(\bar{3}, \bar{4}, \bar{0})\bar{2}1 = \bar{1}2$
- $((2, 3, 4)(\bar{2}, \bar{3}, \bar{4})\bar{1}0)^{\{(1,\bar{3},4,\bar{0})(2,\bar{2})(3,\bar{4},0,\bar{1})\}} = (\bar{0}1)^{\{(1,\bar{3},4,\bar{0})(2,\bar{2})(3,\bar{4},0,\bar{1})\}}$
  $\Rightarrow (2, 4, 0)(\bar{2}, \bar{4}, \bar{0})3\bar{1} = 1\bar{3}$

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0})\}} = (\bar{0}1)^{\{(1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0})\}}$
  $\Rightarrow (2,1,0)(\bar{2},\bar{1},\bar{0})\bar{3}4 = \bar{4}3$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{0},3,\bar{2})(2,\bar{1},0,\bar{3})(4,\bar{4})\}} = (\bar{0}1)^{\{(1,\bar{0},3,\bar{2})(2,\bar{1},0,\bar{3})(4,\bar{4})\}}$
  $\Rightarrow (1,2,4)(\bar{1},\bar{2},\bar{4})0\bar{3} = 3\bar{0}$

From the relations above, we have
$012\bar{3} = 0\bar{1}\underline{\bar{1}2}\bar{3} = 0\bar{1}(3,4,0)(\bar{3},\bar{4},\bar{0})\bar{2}1\bar{3} = (3,4,0)(\bar{3},\bar{4},\bar{0})3\bar{1}\bar{2}\underline{1\bar{3}}$
$= (3,4,0)(\bar{3},\bar{4},\bar{0})3\bar{1}\bar{2}(2,4,0)(\bar{2},\bar{4},\bar{0})3\bar{1} = (3,0)(4,2)(\bar{3},\bar{0})(\bar{4},\bar{2})3\bar{1}\underline{\bar{4}3}\bar{1}$
$= (3,0)(4,2)(\bar{3},\bar{0})(\bar{4},\bar{2})3\bar{1}(2,1,0)(\bar{2},\bar{1},\bar{0})\bar{3}4\bar{1} = (3,2,4,1,0)(\bar{3},\bar{2},\bar{4},\bar{1},\bar{0})\underline{3\bar{0}}\bar{3}4\bar{1}$
$= (3,2,4,1,0)(\bar{3},\bar{2},\bar{4},\bar{1},\bar{0})(1,2,4)(\bar{1},\bar{2},\bar{4})0\bar{3}\bar{3}4\bar{1} = (3,4,2,1,0)(\bar{3},\bar{4},\bar{2},\bar{1},\bar{0})034\bar{1}.$ □

Since, $N^{(012\bar{3})} \geq N^{012\bar{3}} =< e >$, and
$N(t_0t_1t_2\bar{t_3})^{\{(1,3)(2,4)(\bar{1},\bar{3})(\bar{2},\bar{4})\}} = Nt_0t_3t_4\bar{t_1} \Rightarrow (1,3)(2,4)(\bar{1},\bar{3})(\bar{2},\bar{4}) \in N^{(012\bar{3})}$.
$\Rightarrow N^{(012\bar{3})} =< (1,3)(2,4)(\bar{1},\bar{3})(\bar{2},\bar{4}) >= \{e, (1,3)(2,4)(\bar{1},\bar{3})(\bar{2},\bar{4})\}$.

Therefore, the number of single cosets in double coset $[012\bar{3}]$ are at most

$\frac{|N|}{|N^{(012\bar{3})}|} = \frac{5!}{2} = \frac{120}{2} = 60$. Each single coset has two different names.

By conjugating $(012\bar{3} \sim 034\bar{1})$ with the elements of transversal of $N^{(012\bar{3})}$, we obtain the sixty single cosets of double coset $[012\bar{3}]$:
$012\bar{3} \sim 034\bar{1}, 123\bar{4} \sim 140\bar{2}, 234\bar{0} \sim 201\bar{3}, 340\bar{1} \sim 312\bar{4}, 124\bar{0} \sim 103\bar{2}, 401\bar{2} \sim 423\bar{0},$
$240\bar{3} \sim 231\bar{4}, 243\bar{1} \sim 210\bar{4}, 230\bar{1} \sim 214\bar{3}, 412\bar{0} \sim 403\bar{1}, 431\bar{0} \sim 402\bar{3}, 301\bar{4} \sim 342\bar{0},$
$430\bar{2} \sim 421\bar{3}, 304\bar{2} \sim 321\bar{0}, 341\bar{2} \sim 320\bar{2}, 024\bar{3} \sim 031\bar{2}, 324\bar{1} \sim 310\bar{2}, 014\bar{2} \sim 023\bar{1},$
$302\bar{1} \sim 314\bar{0}, 013\bar{4} \sim 042\bar{1}, 041\bar{3} \sim 032\bar{4}, 410\bar{3} \sim 432\bar{1}, 143\bar{0} \sim 102\bar{4}, 142\bar{3} \sim 130\bar{4},$
$021\bar{4} \sim 043\bar{2}, 134\bar{2} \sim 120\bar{3}, 413\bar{2} \sim 420\bar{1}, 213\bar{0} \sim 204\bar{1}, 203\bar{4} \sim 241\bar{0}, 132\bar{0} \sim 104\bar{3},$
$\bar{0}\bar{1}\bar{2}4 \sim \bar{0}\bar{4}\bar{3}1, \bar{1}\bar{2}\bar{4}3 \sim \bar{1}\bar{3}\bar{0}2, \bar{1}\bar{2}\bar{3}0 \sim \bar{1}\bar{0}\bar{4}2, \bar{2}\bar{4}\bar{3}0 \sim \bar{2}\bar{0}\bar{1}4, \bar{2}\bar{3}\bar{0}4 \sim \bar{2}\bar{4}\bar{1}3, \bar{2}\bar{3}\bar{4}1 \sim \bar{2}\bar{1}\bar{0}3,$
$\bar{4}\bar{1}\bar{2}3 \sim \bar{4}\bar{3}\bar{0}1, \bar{3}\bar{0}\bar{4}1 \sim \bar{3}\bar{1}\bar{2}0, \bar{3}\bar{4}\bar{1}0 \sim \bar{3}\bar{0}\bar{2}4, \bar{3}\bar{4}\bar{0}2 \sim \bar{3}\bar{2}\bar{1}4, \bar{3}\bar{0}\bar{1}2 \sim \bar{3}\bar{2}\bar{4}0, \bar{0}\bar{2}\bar{3}4 \sim \bar{0}\bar{4}\bar{1}2,$
$\bar{4}\bar{1}\bar{0}2 \sim \bar{4}\bar{2}\bar{3}1, \bar{4}\bar{0}\bar{2}1 \sim \bar{4}\bar{1}\bar{3}0, \bar{4}\bar{0}\bar{1}3 \sim \bar{4}\bar{3}\bar{2}0, \bar{2}\bar{4}\bar{0}1 \sim \bar{2}\bar{1}\bar{3}4, \bar{1}\bar{3}\bar{4}0 \sim \bar{1}\bar{0}\bar{2}3, \bar{0}\bar{2}\bar{1}3 \sim \bar{0}\bar{3}\bar{4}2,$
$\bar{0}\bar{1}\bar{3}2 \sim \bar{0}\bar{2}\bar{4}1, \bar{4}\bar{0}\bar{3}2 \sim \bar{4}\bar{2}\bar{1}0, \bar{4}\bar{3}\bar{1}2 \sim \bar{4}\bar{2}\bar{0}3, \bar{1}\bar{3}\bar{2}4 \sim \bar{1}\bar{4}\bar{0}3, \bar{0}\bar{1}\bar{4}3 \sim \bar{0}\bar{3}\bar{2}1, \bar{0}\bar{3}\bar{1}4 \sim \bar{0}\bar{4}\bar{2}3,$
$\bar{3}\bar{1}\bar{0}4 \sim \bar{3}\bar{4}\bar{2}1, \bar{1}\bar{4}\bar{3}2 \sim \bar{1}\bar{2}\bar{0}4, \bar{3}\bar{1}\bar{4}2 \sim \bar{3}\bar{2}\bar{0}1, \bar{1}\bar{4}\bar{2}0 \sim \bar{1}\bar{0}\bar{3}4, \bar{2}\bar{1}\bar{4}0 \sim \bar{2}\bar{0}\bar{3}1, \bar{2}\bar{3}\bar{1}0 \sim \bar{2}\bar{0}\bar{4}3.$

The orbits of $N^{(012\bar{3})}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{0\}, \{\bar{0}\}, \{1,3\}, \{2,4\}, \{\bar{1},\bar{3}\}$ and $\{\bar{2},\bar{4}\}$. Now, take one representative from each orbit namely $0, \bar{0}, 3, 4$, $\bar{3}$ and $\bar{4}$, then find the double cosets to which $Nt_0t_1t_2\bar{t_3}t_0$, $Nt_0t_1t_2\bar{t_3}\bar{t_0}$, $Nt_0t_1t_2\bar{t_3}t_3$, $Nt_0t_1t_2\bar{t_3}\bar{t_3}$, $Nt_0t_1t_2\bar{t_3}t_4$ and $Nt_0t_1t_2\bar{t_3}\bar{t_4}$ belong. However,

$Nt_0t_1t_2\bar{t_3}t_0 = Nt_0t_1t_2\bar{t_3}t_0 \in [0\bar{1}\bar{2}3]$ **(By Lemma-31).**
$Nt_0t_1t_2\bar{t_3}\bar{t_0} = Nt_0t_1t_2\bar{t_3}\bar{t_0} \in Nt_0t_1t_2\bar{t_3}\bar{t_0}N$ (New double coset denote by $[012\bar{3}\bar{0}]$).
$Nt_0t_1t_2\bar{t_3}t_3 = Nt_0t_1t_2 \in [012]$.
$Nt_0t_1t_2\bar{t_3}\bar{t_3} = Nt_0t_1t_2t_3 \in [0123]$.
$Nt_0t_1t_2\bar{t_3}t_4 = Nt_0t_1t_2\bar{t_3}t_4 \in [012]$ **(By Lemma-32).**
$Nt_0t_1t_2\bar{t_3}\bar{t_4} = Nt_0t_1t_2\bar{t_3}\bar{t_4} \in [0123]$ **(By Lemma-33).**

Consider the double coset $[01\bar{2}\bar{0}]$
Claim: $01\bar{2}\bar{0} \sim \bar{0}\bar{3}20$

Proof. By our relation-I, $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtain the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,0,2)(\bar{1},\bar{0},\bar{2})\}} = (\bar{0}1)^{\{(1,0,2)(\bar{1},\bar{0},\bar{2})\}}$
  $\Rightarrow (1,3,4)(\bar{1},\bar{3},\bar{4})\bar{0}2 = \bar{2}0$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{0},3,\bar{2})(2,\bar{1},0,\bar{3})(4,\bar{4})\}} = (\bar{0}1)^{\{(1,\bar{0},3,\bar{2})(2,\bar{1},0,\bar{3})(4,\bar{4})\}}$
  $\Rightarrow (1,2,4)(\bar{1},\bar{2},\bar{4})0\bar{3} = 3\bar{0}$

From the relations above, we have
$01\bar{2}\bar{0} = 01\underline{\bar{2}0}0 = 01(1,3,4)(\bar{1},\bar{3},\bar{4})\bar{0}20 = (1,3,4)(\bar{1},\bar{3},\bar{4})0\underline{3\bar{0}}20$
$= (1,3,4)(\bar{1},\bar{3},\bar{4})0(1,2,4)(\bar{1},\bar{2},\bar{4})0\bar{3}20 = (1,3)(4,2)(\bar{1},\bar{3})(\bar{4},\bar{2})\bar{0}\bar{3}20.$ □

Since, $N^{(01\bar{2}\bar{0})} \geq N^{01\bar{2}\bar{0}} =< e >$, and
$N(t_0t_1\bar{t_2}\bar{t_0})^{\{(1,\bar{3})(2,\bar{2})(3,\bar{1})(4,\bar{4})(0,\bar{0})\}} = \bar{0}\bar{3}20 \Rightarrow (1,\bar{3})(2,\bar{2})(3,\bar{1})(4,\bar{4})(0,\bar{0}) \in N^{(01\bar{2}\bar{0})}$
$\Rightarrow N^{(01\bar{2}\bar{0})} =< (1,\bar{3})(2,\bar{2})(3,\bar{1})(4,\bar{4})(0,\bar{0}) >= \{e,(1,\bar{3})(2,\bar{2})(3,\bar{1})(4,\bar{4})(0,\bar{0})\}$.

Therefore, the number of single cosets in the double coset $[01\bar{2}\bar{0}]$ are at most

$$\frac{|N|}{|N^{(0120)}|}=\frac{5!}{2}=\frac{120}{2}=60.\ \text{Each single coset has two different names.}$$

By conjugating $(01\bar{2}\bar{0} \sim \bar{0}\bar{3}20)$ with the elements of transversal of $N^{(01\bar{2}\bar{0})}$, we obtain the sixty single cosets of double coset $[01\bar{2}\bar{0}]$:

$01\bar{2}\bar{0} \sim \bar{0}\bar{3}20, 12\bar{3}\bar{1} \sim \bar{1}\bar{4}31, 04\bar{2}\bar{0} \sim \bar{0}\bar{1}20, 23\bar{4}\bar{2} \sim \bar{2}\bar{0}42, \bar{1}\bar{2}41 \sim 13\bar{4}\bar{1}, 10\bar{3}\bar{1} \sim \bar{1}\bar{2}31,$
$34\bar{0}\bar{3} \sim \bar{3}\bar{1}03, \bar{2}\bar{4}32 \sim 20\bar{3}\bar{2}, \bar{2}\bar{3}02 \sim 24\bar{0}\bar{2}, 21\bar{4}\bar{2} \sim \bar{2}\bar{3}42, \bar{1}\bar{0}41 \sim 12\bar{4}\bar{1}, 40\bar{1}\bar{4} \sim \bar{4}\bar{2}14,$
$\bar{4}\bar{3}04 \sim 41\bar{0}\bar{4}, \bar{3}\bar{0}43 \sim 31\bar{4}\bar{3}, \bar{3}\bar{4}13 \sim 30\bar{1}\bar{3}, 32\bar{0}\bar{3} \sim \bar{3}\bar{4}03, \bar{2}\bar{1}32 \sim 24\bar{3}\bar{2}, \bar{2}\bar{1}02 \sim 23\bar{0}\bar{2};$
$\bar{3}\bar{0}13 \sim 32\bar{1}\bar{3}, \bar{0}\bar{4}10 \sim 02\bar{1}\bar{0}, \bar{4}\bar{1}04 \sim 42\bar{0}\bar{4}, 40\bar{3}\bar{4} \sim \bar{4}\bar{1}34, \bar{4}\bar{0}24 \sim 41\bar{2}\bar{4}, 43\bar{1}\bar{4} \sim \bar{4}\bar{0}14,$
$\bar{4}\bar{2}04 \sim 43\bar{0}\bar{4}, \bar{3}\bar{2}43 \sim 30\bar{4}\bar{3}; \bar{3}\bar{2}13 \sim 34\bar{1}\bar{3}, \bar{2}\bar{4}02 \sim 21\bar{0}\bar{2}, \bar{4}\bar{1}24 \sim 43\bar{2}\bar{4}, \bar{1}\bar{0}21 \sim 13\bar{2}\bar{1},$
$\bar{0}\bar{2}10 \sim 03\bar{1}\bar{0}, 31\bar{0}\bar{3} \sim \bar{3}\bar{2}03, 01\bar{4}\bar{0} \sim \bar{0}\bar{2}40, \bar{0}\bar{1}30 \sim 02\bar{3}\bar{0}, 30\bar{2}\bar{3} \sim \bar{3}\bar{1}23, \bar{0}\bar{3}10 \sim 04\bar{1}\bar{0},$
$\bar{4}\bar{0}34 \sim 42\bar{3}\bar{4}, \bar{4}\bar{3}24 \sim 40\bar{2}\bar{4}, 42\bar{1}\bar{4} \sim \bar{4}\bar{3}14, \bar{0}\bar{2}30 \sim 04\bar{3}\bar{0}, \bar{3}\bar{4}23 \sim 31\bar{2}\bar{3}, \bar{1}\bar{4}21 \sim 10\bar{2}\bar{1},$
$\bar{1}\bar{3}21 \sim 14\bar{2}\bar{1}, 12\bar{0}\bar{1} \sim \bar{1}\bar{3}01, 41\bar{3}\bar{4} \sim \bar{4}\bar{2}34, \bar{0}\bar{1}40 \sim 03\bar{4}\bar{0}, \bar{0}\bar{4}30 \sim 01\bar{3}\bar{0}, 34\bar{2}\bar{3} \sim \bar{3}\bar{0}23,$
$03\bar{2}\bar{0} \sim \bar{0}\bar{4}20, \bar{1}\bar{3}41 \sim 10\bar{4}\bar{1}, 02\bar{4}\bar{0} \sim \bar{0}\bar{3}40, \bar{2}\bar{0}32 \sim 21\bar{3}\bar{2}, 23\bar{1}\bar{2} \sim \bar{2}\bar{4}12, \bar{1}\bar{2}01 \sim 14\bar{0}\bar{1},$
$\bar{3}\bar{1}43 \sim 32\bar{4}\bar{3}, 14\bar{3}\bar{1} \sim \bar{1}\bar{0}31, 13\bar{0}\bar{1} \sim \bar{1}\bar{4}01, 20\bar{4}\bar{2} \sim \bar{2}\bar{1}42, \bar{2}\bar{3}12 \sim 20\bar{1}\bar{2}, 24\bar{1}\bar{2} \sim \bar{2}\bar{0}12.$

The orbits of $N^{(01\bar{2}\bar{0})}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{1,\bar{3}\}, \{2,\bar{2}\}, \{3,\bar{1}\}, \{4,\bar{4}\}$ and $\{0,\bar{0}\}$. Now, take one representative from each orbit, namely $1, 2, 3, 4$ and 0, then find the double cosets to which $Nt_0t_1\bar{t_2}\bar{t_0}t_1$, $Nt_0t_1\bar{t_2}\bar{t_0}t_2$, $Nt_0t_1\bar{t_2}\bar{t_0}t_3$, $Nt_0t_1\bar{t_2}\bar{t_0}t_4$ and $Nt_0t_1\bar{t_2}\bar{t_0}t_0$ belong. However,

$Nt_0t_1\bar{t_2}\bar{t_0}t_1 = Nt_0t_1\bar{t_2}\bar{t_0}t_1 \in [0123]$ **(By Lemma-34).**
$Nt_0t_1\bar{t_2}\bar{t_0}t_2 = Nt_0t_1\bar{t_2}\bar{t_0}t_2 \in [0120]$ **(By Lemma-35).**
$Nt_0t_1\bar{t_2}\bar{t_0}t_3 = Nt_0t_1\bar{t_2}\bar{t_0}t_3 \in [0120\bar{3}]$ **(By Lemma-36).**
$Nt_0t_1\bar{t_2}\bar{t_0}t_4 = Nt_0t_1\bar{t_2}\bar{t_0}t_4 \in [012\bar{3}\bar{0}]$ **(By Lemma-37).**
$Nt_0t_1\bar{t_2}\bar{t_0}t_0 = Nt_0t_1\bar{t_2}\bar{t_0}t_0 \in [01\bar{2}]$.

Consider the double coset $[01\bar{2}\bar{3}]$
Claim: $01\bar{2}\bar{3} \sim \bar{2}\bar{0}43 \sim \bar{1}\bar{4}03$

Proof. By our relation-I, $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtained the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}} = (\bar{0}1)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}}$
  $\Rightarrow (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0} = 0\bar{1}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0})\}} = (\bar{0}1)^{\{(1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0})\}}$
  $\Rightarrow (2,1,0)(\bar{2},\bar{1},\bar{0})\bar{3}4 = \bar{4}3$

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,3)(0,2)(\bar{1},\bar{3})(\bar{0},\bar{2})\}} = (\bar{0}1)^{\{(1,3)(0,2)(\bar{1},\bar{3})(\bar{0},\bar{2})\}}$
  $\Rightarrow (0,1,4)(\bar{0},\bar{1},\bar{4})\bar{3}2 = \bar{2}3$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{3},0,\bar{4})(2,\bar{2})(3,\bar{0},4,\bar{1})\}} = (\bar{0}1)^{\{(1,\bar{3},0,\bar{4})(2,\bar{2})(3,\bar{0},4,\bar{1})\}}$
  $\Rightarrow (2,0,1)(\bar{2},\bar{0},\bar{1})3\bar{4} = 4\bar{3}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,2,3,0,4)(\bar{1},\bar{2},\bar{3},\bar{0},\bar{4})\}} = (\bar{0}1)^{\{(1,2,3,0,4)(\bar{1},\bar{2},\bar{3},\bar{0},\bar{4})\}}$
  $\Rightarrow (3,0,1)(\bar{3},\bar{0},\bar{1})\bar{2}4 = \bar{4}2$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{2})(2,\bar{1})(3,\bar{3})(4,\bar{4})(0,\bar{0})\}} = (\bar{0}1)^{\{(1,\bar{2})(2,\bar{1})(3,\bar{3})(4,\bar{4})(0,\bar{0})\}}$
  $\Rightarrow (1,3,4)(\bar{1},\bar{3},\bar{4})2\bar{0} = 0\bar{2}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,3,2)(\bar{1},\bar{3},\bar{2})\}} = (\bar{0}1)^{\{(1,3,2)(\bar{1},\bar{3},\bar{2})\}}$
  $\Rightarrow (1,2,4)(\bar{1},\bar{2},\bar{4})\bar{3}0 = \bar{0}3$

From the relation's above, we have
1. $01\bar{2}\bar{3} = 01\underline{\bar{2}3}3 = 01(0,1,4)(\bar{0},\bar{1},\bar{4})\bar{3}23 = (0,1,4)(\bar{0},\bar{1},\bar{4})1\underline{4\bar{3}}23$
$= (0,1,4)(\bar{0},\bar{1},\bar{4})1(2,0,1)(\bar{2},\bar{0},\bar{1})3\bar{4}23 = (0,2)(4,1)(\bar{0},\bar{2})(\bar{4},\bar{1})23\underline{\bar{4}2}3$
$= (0,2)(4,1)(\bar{0},\bar{2})(\bar{4},\bar{1})23(3,0,1)(\bar{3},\bar{0},\bar{1})\bar{2}43 = (0,2,1,4,3)(\bar{0},\bar{2},\bar{1},\bar{4},\bar{3})2\underline{0\bar{2}}43$
$= (0,2,1,4,3)(\bar{0},\bar{2},\bar{1},\bar{4},\bar{3})2(1,3,4)(\bar{1},\bar{3},\bar{4})2\bar{0}43 = (0,2,3)(\bar{0},\bar{2},\bar{3})22\bar{0}43$
$= (0,2,3)(\bar{0},\bar{2},\bar{3})\bar{2}\bar{0}43.$

2. $01\bar{2}\bar{3} = \bar{0}\underline{\bar{0}1}\bar{2}\bar{3} = \bar{0}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0\bar{2}\bar{3} = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}\underline{0\bar{2}}\bar{3}$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}(1,3,4)(\bar{1},\bar{3},\bar{4})2\bar{0}\bar{3} = (2,4)(3,1)(\bar{2},\bar{4})(\bar{3},\bar{1})\bar{0}\bar{3}2\bar{0}\bar{3}$
$= (2,4)(3,1)(\bar{2},\bar{4})(\bar{3},\bar{1})\bar{0}\bar{3}2\underline{\bar{0}\bar{3}}3 = (2,4)(3,1)(\bar{2},\bar{4})(\bar{3},\bar{1})\bar{0}\bar{3}2(1,2,4)(\bar{1},\bar{2},\bar{4})\bar{3}03$
$= (2,1,3)(\bar{2},\bar{1},\bar{3})\bar{0}\underline{\bar{3}4}\bar{3}03 = (2,1,3)(\bar{2},\bar{1},\bar{3})\bar{0}(2,1,3)(\bar{2},\bar{0},\bar{1})\bar{4}3\bar{3}03$
$= (1,3,0)(\bar{1},\bar{3},\bar{0})\bar{1}\bar{4}03.$ □

Since, $N^{(01\bar{2}\bar{3})} \geq N^{01\bar{2}\bar{3}} =< e >$, and
$N(t_0t_1\bar{t_2}\bar{t_3})^{\{(1,\bar{0},2,\bar{4})(3,\bar{3})(4,\bar{1},0,\bar{2})\}} = \bar{2}\bar{0}43 \Rightarrow (1,\bar{0},2,\bar{4})(3,\bar{3})(4,\bar{1},0,\bar{2}) \in N^{(01\bar{2}\bar{3})}$
$N(t_0t_1\bar{t_2}\bar{t_3})^{\{(1,\bar{4},2,\bar{0})(4,\bar{2})(4,\bar{1},0,\bar{1})\}} = \bar{1}\bar{4}03 \Rightarrow (1,\bar{4},2,\bar{0})(4,\bar{2})(4,\bar{1},0,\bar{1}) \in N^{(01\bar{2}\bar{3})}$
$\Rightarrow N^{(01\bar{2}\bar{3})} =< (1,\bar{0},2,\bar{4})(3,\bar{3})(4,\bar{1},0,\bar{2}), (1,\bar{4},2,\bar{0})(4,\bar{2})(4,\bar{1},0,\bar{1}) >$
$= \{e, (1,\bar{0},2,\bar{4})(3,\bar{3})(4,\bar{1},0,\bar{2}), (1,\bar{4},2,\bar{0})(3,\bar{3})(4,\bar{2},0,\bar{1}), (1,2)(4,0)(\bar{1},\bar{2})(\bar{4},\bar{0})\}.$

Now, take $Nt_0t_1\bar{t_2}\bar{t_3}$ conjugates with elements of $N^{(01\bar{2}\bar{3})}$, we obtain the following relations:

$$01\bar{2}\bar{3} \sim 42\bar{1}\bar{3} \sim \bar{2}\bar{0}43 \sim \bar{1}\bar{4}03$$

Therefore, the number of single cosets in the double coset $[01\bar{2}\bar{3}]$ are at most

$\frac{|N|}{|N^{(01\bar{2}\bar{3})}|}=\frac{5!}{4}=\frac{120}{4}=30$. Each single coset has four different names.

By conjugating $01\bar{2}\bar{3} \sim 42\bar{1}\bar{3} \sim \bar{2}\bar{0}43 \sim \bar{1}\bar{4}03$ with the elements of transversal of $N^{(01\bar{2}\bar{3})}$, we obtain the following thirty single cosets of double coset $[01\bar{2}\bar{3}]$:

1. $01\bar{2}\bar{3} \sim 42\bar{1}\bar{3} \sim \bar{2}\bar{0}43 \sim \bar{1}\bar{4}03$
2. $12\bar{3}\bar{4} \sim 03\bar{2}\bar{4} \sim \bar{3}\bar{1}04 \sim \bar{2}\bar{0}14$
3. $\bar{0}\bar{1}24 \sim \bar{3}\bar{2}14 \sim 20\bar{3}\bar{4} \sim 13\bar{0}\bar{4}$
4. $23\bar{4}\bar{0} \sim 14\bar{3}\bar{0} \sim \bar{4}\bar{2}10 \sim \bar{3}\bar{1}20$
5. $\bar{1}\bar{2}43 \sim \bar{0}\bar{4}23 \sim 41\bar{0}\bar{3} \sim 20\bar{1}\bar{3}$
6. $\bar{1}\bar{2}30 \sim \bar{4}\bar{3}20 \sim 31\bar{4}\bar{0} \sim 24\bar{1}\bar{0}$
7. $20\bar{4}\bar{1} \sim 34\bar{0}\bar{1} \sim \bar{4}\bar{2}31 \sim \bar{0}\bar{3}21$
8. $\bar{2}\bar{4}30 \sim \bar{1}\bar{3}40 \sim 32\bar{1}\bar{0} \sim 41\bar{2}\bar{0}$
9. $\bar{2}\bar{3}04 \sim \bar{1}\bar{0}34 \sim 02\bar{1}\bar{4} \sim 31\bar{2}\bar{4}$
10. $\bar{0}\bar{4}31 \sim \bar{2}\bar{3}41 \sim 30\bar{2}\bar{1} \sim 42\bar{0}\bar{1}$
11. $12\bar{4}\bar{0} \sim 34\bar{2}\bar{0} \sim \bar{4}\bar{1}30 \sim \bar{2}\bar{3}10$
12. $40\bar{1}\bar{2} \sim 31\bar{0}\bar{2} \sim \bar{1}\bar{4}32 \sim \bar{0}\bar{3}42$
13. $\bar{2}\bar{0}31 \sim \bar{4}\bar{3}03 \sim 32\bar{4}\bar{1} \sim 04\bar{2}\bar{1}$
14. $\bar{2}\bar{4}01 \sim \bar{3}\bar{0}41 \sim 02\bar{3}\bar{1} \sim 43\bar{2}\bar{1}$
15. $\bar{2}\bar{1}40 \sim \bar{3}\bar{4}10 \sim 42\bar{3}\bar{0} \sim 13\bar{2}\bar{0}$
16. $\bar{4}\bar{1}23 \sim \bar{0}\bar{2}13 \sim 24\bar{0}\bar{3} \sim 10\bar{4}\bar{3}$
17. $\bar{1}\bar{0}42 \sim \bar{3}\bar{4}02 \sim 41\bar{3}\bar{2} \sim 03\bar{1}\bar{2}$
18. $03\bar{4}\bar{1} \sim 24\bar{3}\bar{1} \sim \bar{4}\bar{0}21 \sim \bar{3}\bar{2}01$
19. $\bar{3}\bar{4}21 \sim \bar{0}\bar{2}41 \sim 23\bar{0}\bar{1} \sim 40\bar{3}\bar{1}$
20. $\bar{3}\bar{0}12 \sim \bar{4}\bar{1}02 \sim 13\bar{4}\bar{2} \sim 04\bar{3}\bar{2}$
21. $43\bar{0}\bar{2} \sim 10\bar{3}\bar{2} \sim \bar{0}\bar{4}12 \sim \bar{3}\bar{1}42$
22. $21\bar{3}\bar{0} \sim 43\bar{1}\bar{0} \sim \bar{3}\bar{2}40 \sim \bar{1}\bar{4}20$
23. $30\bar{1}\bar{4} \sim 21\bar{0}\bar{4} \sim \bar{1}\bar{3}24 \sim \bar{0}\bar{2}34$
24. $\bar{4}\bar{0}13 \sim \bar{2}\bar{1}03 \sim 14\bar{2}\bar{3} \sim 02\bar{4}\bar{3}$
25. $14\bar{0}\bar{2} \sim 30\bar{4}\bar{2} \sim \bar{0}\bar{1}32 \sim \bar{4}\bar{3}12$
26. $\bar{4}\bar{0}32 \sim \bar{1}\bar{3}02 \sim 34\bar{1}\bar{3} \sim 01\bar{4}\bar{2}$

27. $04\bar{1}\bar{3} \sim 21\bar{4}\bar{3} \sim \bar{1}\bar{0}23 \sim \bar{4}\bar{2}03$
28. $40\bar{2}\bar{3} \sim 12\bar{0}\bar{3} \sim \bar{2}\bar{0}13 \sim \bar{0}\bar{1}43$
29. $32\bar{0}\bar{4} \sim 10\bar{2}\bar{4} \sim \bar{0}\bar{3}14 \sim \bar{2}\bar{1}34$
30. $\bar{1}\bar{2}04 \sim \bar{3}\bar{0}24 \sim 01\bar{3}\bar{4} \sim 23\bar{1}\bar{4}$

The orbits of $N^{(01\bar{2}\bar{3})}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{3,\bar{3}\}$, $\{1,2,\bar{4},\bar{0}\}$ and $\{4,0,\bar{1},\bar{2}\}$. Now, take one representative from each orbit, namely $3, 1$ and $\bar{1}$, then find the double coset to which $Nt_0t_1\bar{t_2}\bar{t_3}t_3$, $Nt_0t_1\bar{t_2}\bar{t_3}t_1$ and $Nt_0t_1\bar{t_2}\bar{t_3}\bar{t_1}$ belong. However,

$Nt_0t_1\bar{t_2}\bar{t_3}t_3 = Nt_0t_1\bar{t_2} \in [01\bar{2}]$.
$Nt_0t_1\bar{t_2}\bar{t_3}t_1 = Nt_0t_1\bar{t_2}\bar{t_3}t_1 \in [0123]$ **(By Lemma-38).**
$Nt_0t_1\bar{t_2}\bar{t_3}\bar{t_1} = Nt_0t_1\bar{t_2}\bar{t_3}\bar{t_1} \in [0120\bar{3}]$ **(By Lemma-39).**

Consider the double coset $[0\bar{1}\bar{2}3]$
Claim: $0\bar{1}\bar{2}3 \sim 4\bar{1}\bar{3}2$

Proof. By our relation-I, $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtain the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,3)(0,2)(\bar{1},\bar{3})(\bar{0},\bar{2})\}} = (\bar{0}1)^{\{(1,3)(0,2)(\bar{1},\bar{3})(\bar{0},\bar{2})\}}$
  $\Rightarrow (0,1,4)(\bar{0},\bar{1},\bar{4})\bar{3}2 = \bar{2}3$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{4},2,\bar{0})(3,\bar{3})(4,\bar{2},0,\bar{1})\}} = (\bar{0}1)^{\{(1,\bar{4},2,\bar{0})(3,\bar{3})(4,\bar{2},0,\bar{1})\}}$
  $\Rightarrow (0,3,2)(\bar{0},\bar{3},\bar{2})4\bar{1} = 1\bar{4}$

From the relations above, we have
$0\bar{1}\underline{\bar{2}3} = 0\bar{1}(0,1,4)(\bar{0},\bar{1},\bar{4})\bar{3}2 = (0,1,4)(\bar{0},\bar{1},\bar{4})\underline{1\bar{4}}\bar{3}2 = (0,1,4)(\bar{0},\bar{1},\bar{4})(0,3,2)(\bar{0},\bar{3},\bar{2})4\bar{1}\bar{3}2$
$= (0,1,4,3,2)(\bar{0},\bar{1},\bar{4},\bar{3},\bar{2})4\bar{1}\bar{3}2.$ □

Since, $N^{(0\bar{1}\bar{2}3)} \geq N^{0\bar{1}\bar{2}3} =< e >$, and
$N(t_0\bar{t_1}\bar{t_2}t_3)^{\{(2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0})\}} = 4\bar{1}\bar{3}2 \Rightarrow (2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0}) \in N^{(0\bar{1}\bar{2}3)}$
$\Rightarrow N^{(0\bar{1}\bar{2}3)} =< (2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0}) >= \{e, (2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0})\}$.

Therefore, the number of single cosets in the double coset $[0\bar{1}\bar{2}3]$ are at most

$\frac{|N|}{|N^{(0123)}|} = \frac{5!}{2} = \frac{120}{2} = 60$. Each single coset has two different names.

By conjugating $(0\bar{1}\bar{2}3 \sim 4\bar{1}\bar{3}2)$ with the elements of transversal of $N^{(0\bar{1}\bar{2}3)}$, we obtain the following sixty single cosets of double coset $[0\bar{1}\bar{2}3]$:
$0\bar{1}\bar{2}3 \sim 4\bar{1}\bar{3}2, 1\bar{2}\bar{3}4 \sim 0\bar{2}\bar{4}3, \bar{0}12\bar{4} \sim \bar{3}14\bar{2}, 2\bar{3}\bar{4}0 \sim 1\bar{3}\bar{0}4, \bar{1}24\bar{3} \sim \bar{0}23\bar{4}, \bar{1}23\bar{0} \sim \bar{4}20\bar{3},$
$3\bar{4}\bar{2}1 \sim 2\bar{4}\bar{1}0, \bar{2}43\bar{0} \sim \bar{1}40\bar{3}, \bar{2}30\bar{4} \sim \bar{1}34\bar{0}, \bar{2}34\bar{1} \sim \bar{0}31\bar{4}, 1\bar{2}\bar{4}0 \sim 3\bar{2}\bar{0}4, 4\bar{0}\bar{1}2 \sim 3\bar{0}\bar{2}1,$
$\bar{4}30\bar{1} \sim \bar{2}31\bar{0}, \bar{3}04\bar{1} \sim \bar{2}01\bar{4}, \bar{2}40\bar{1} \sim \bar{3}41\bar{0}, 2\bar{4}\bar{0}3 \sim 1\bar{4}\bar{3}0, \bar{3}40\bar{2} \sim \bar{1}42\bar{0}, 2\bar{4}\bar{3}1 \sim 0\bar{4}\bar{1}3,$
$2\bar{3}\bar{0}1 \sim 4\bar{3}\bar{1}0, \bar{3}01\bar{2} \sim \bar{4}02\bar{1}, \bar{0}41\bar{2} \sim \bar{3}42\bar{1}, \bar{4}10\bar{2} \sim \bar{3}12\bar{0}, 4\bar{0}\bar{3}1 \sim 2\bar{0}\bar{1}3, 3\bar{0}\bar{1}4 \sim 2\bar{0}\bar{4}1,$
$\bar{4}01\bar{3} \sim \bar{2}03\bar{1}, 4\bar{3}\bar{0}2 \sim 1\bar{3}\bar{2}0, 3\bar{0}\bar{4}2 \sim 1\bar{0}\bar{2}4, 3\bar{4}\bar{1}2 \sim 0\bar{4}\bar{2}1, \bar{0}13\bar{2} \sim \bar{4}12\bar{3}, \bar{1}02\bar{3} \sim \bar{4}03\bar{2},$
$0\bar{3}\bar{1}2 \sim 4\bar{3}\bar{2}1, \bar{0}21\bar{3} \sim \bar{4}23\bar{1}, 4\bar{1}\bar{2}0 \sim 3\bar{1}\bar{0}2, 0\bar{1}\bar{4}2 \sim 3\bar{1}\bar{2}4, 4\bar{1}\bar{0}3 \sim 2\bar{1}\bar{3}0, 4\bar{0}\bar{2}3 \sim 1\bar{0}\bar{3}2,$
$\bar{4}31\bar{2} \sim \bar{0}32\bar{1}, \bar{0}14\bar{3} \sim \bar{2}13\bar{4}, 1\bar{4}\bar{2}3 \sim 0\bar{4}\bar{3}2, \bar{1}32\bar{4} \sim \bar{0}34\bar{2}, 3\bar{2}\bar{4}1 \sim 0\bar{2}\bar{1}4, 0\bar{2}\bar{3}1 \sim 4\bar{2}\bar{3}0,$
$1\bar{2}\bar{0}3 \sim 4\bar{2}\bar{3}0, \bar{3}10\bar{4} \sim \bar{2}14\bar{0}, 0\bar{1}\bar{3}4 \sim 2\bar{1}\bar{4}3, \bar{3}02\bar{4} \sim \bar{1}04\bar{2}, \bar{1}43\bar{2} \sim \bar{0}42\bar{3}, \bar{3}24\bar{0} \sim \bar{1}20\bar{4},$
$2\bar{0}\bar{3}4 \sim 1\bar{0}\bar{4}3, 1\bar{3}\bar{4}2 \sim 0\bar{3}\bar{2}4, \bar{0}24\bar{1} \sim \bar{3}21\bar{4}, 2\bar{3}\bar{1}4 \sim 0\bar{3}\bar{4}1, \bar{3}20\bar{1} \sim \bar{4}21\bar{0}, \bar{4}13\bar{0} \sim \bar{2}10\bar{3},$
$\bar{2}04\bar{3} \sim \bar{1}03\bar{4}, 3\bar{1}\bar{4}0 \sim 2\bar{1}\bar{0}4, \bar{1}30\bar{2} \sim \bar{4}32\bar{0}, 3\bar{4}\bar{2}0 \sim 1\bar{4}\bar{0}2, \bar{2}41\bar{3} \sim \bar{0}43\bar{1}, 4\bar{2}\bar{0}1 \sim 3\bar{2}\bar{1}0.$

The orbits of $N^{(0\bar{1}\bar{2}3)}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{1\}$, $\{\bar{1}\}$, $\{2,3\}$, $\{\bar{2},\bar{3}\}$,$\{4,0\}$ and $\{\bar{4},\bar{0}\}$. Now, take one representative from each orbit, namely 1, $\bar{1}$, 3, $\bar{3}$, 0 and $\bar{0}$, and find the double coset to which $Nt_0\bar{t_1}\bar{t_2}t_3t_1$, $Nt_0\bar{t_1}\bar{t_2}t_3\bar{t_1}$, $Nt_0\bar{t_1}\bar{t_2}t_3t_3$, $Nt_0\bar{t_1}\bar{t_2}t_3\bar{t_3}$, $Nt_0\bar{t_1}\bar{t_2}t_3t_0$ and $Nt_0\bar{t_1}\bar{t_2}t_3\bar{t_0}$ belong. However,

$Nt_0\bar{t_1}\bar{t_2}t_3t_1 = Nt_0\bar{t_1}\bar{t_2}t_3t_1 \in [012\bar{3}\bar{0}]$ **(By Lemma-40)**.
$Nt_0\bar{t_1}\bar{t_2}t_3\bar{t_1} = Nt_0\bar{t_1}\bar{t_2}t_3\bar{t_1} \in [012\bar{3}]$ **(By Lemma-41)**.
$Nt_0\bar{t_1}\bar{t_2}t_3t_3 = Nt_0\bar{t_1}\bar{t_2}\bar{t_3} \in [0\bar{1}\bar{2}\bar{3}]$.
$Nt_0\bar{t_1}\bar{t_2}t_3\bar{t_3} = Nt_0\bar{t_1}\bar{t_2} \in [0\bar{1}\bar{2}]$.
$Nt_0\bar{t_1}\bar{t_2}t_3t_0 = Nt_0\bar{t_1}\bar{t_2}t_3t_0 \in [01023]$ **(By Lemma-42)**.
$Nt_0\bar{t_1}\bar{t_2}t_3\bar{t_0} = Nt_0\bar{t_1}\bar{t_2}t_3\bar{t_0} \in [0102]$ **(By Lemma-43)**.

Consider the double coset $[0\bar{1}\bar{2}\bar{3}]$
Claim: $0\bar{1}\bar{2}\bar{3} \sim 1\bar{0}\bar{3}\bar{2}$

Proof. By our relation-I, $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtain the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}} = (\bar{0}1)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}}$
  $\Rightarrow (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0} = 0\bar{1}$

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{3},0,\bar{2})(2,\bar{1},3,\bar{0})(4,\bar{4})\}} = (\bar{0}1)^{\{(1,\bar{3},0,\bar{2})(2,\bar{1},3,\bar{0})(4,\bar{4})\}}$
  $\Rightarrow (1,0,4)(\bar{1},\bar{0},\bar{4})3\bar{2} = 2\bar{3}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{3},0,\bar{4})(2,\bar{2})(3,\bar{0},4,\bar{1})\}} = (\bar{0}1)^{\{(1,\bar{3},0,\bar{4})(2,\bar{2})(3,\bar{0},4,\bar{1})\}}$
  $\Rightarrow (2,0,1)(\bar{2},\bar{0},\bar{1})3\bar{4} = 4\bar{3}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,2,3,0,4)(\bar{1},\bar{2},\bar{3},\bar{0},\bar{4})\}} = (\bar{0}1)^{\{(1,2,3,0,4)(\bar{1},\bar{2},\bar{3},\bar{0},\bar{4})\}}$
  $\Rightarrow (3,0,1)(\bar{3},\bar{0},\bar{1})\bar{2}4 = \bar{4}2$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,0,4)(\bar{1},\bar{0},\bar{4})\}} = (\bar{0}1)^{\{(1,0,4)(\bar{1},\bar{0},\bar{4})\}}$
  $\Rightarrow (2,3,1)(\bar{2},\bar{3},\bar{1})\bar{0}4 = \bar{4}0$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,3,2)(\bar{1},\bar{3},\bar{2})\}} = (\bar{0}1)^{\{(1,3,2)(\bar{1},\bar{3},\bar{2})\}}$
  $\Rightarrow (1,2,4)(\bar{1},\bar{2},\bar{4})\bar{3}0 = \bar{0}3$

From the relations above, we have
$\underline{0\bar{1}}\bar{2}\bar{3} = (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}\bar{2}\bar{3} = (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}2\underline{2\bar{3}}$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}2(1,0,4)(\bar{1},\bar{0},\bar{4})3\bar{2} = (2,1,0,4,3)(\bar{2},\bar{1},\bar{0},\bar{4},\bar{3})0\underline{\bar{4}2}3\bar{2}$
$= (2,1,0,4,3)(\bar{2},\bar{1},\bar{0},\bar{4},\bar{3})0(3,0,1)(\bar{3},\bar{0},\bar{1})\bar{2}43\bar{2} = (2,3)(0,4)(\bar{2},\bar{3})(\bar{0},\bar{4})1\bar{2}43\bar{2}$
$= (2,3)(0,4)(\bar{2},\bar{3})(\bar{0},\bar{4})1\bar{2}\underline{4\bar{3}}3\bar{2} = (2,3)(0,4)(\bar{2},\bar{3})(\bar{0},\bar{4})1\bar{2}(2,0,1)(\bar{2},\bar{0},\bar{1})3\bar{4}3\bar{2}$
$= (2,3,0,4,1)(\bar{2},\bar{3},\bar{0},\bar{4},\bar{1})2\underline{\bar{0}3}\bar{4}3\bar{2} = (2,3,0,4,1)(\bar{2},\bar{3},\bar{0},\bar{4},\bar{1})2(1,2,4)(\bar{1},\bar{2},\bar{4})\bar{3}0\bar{4}3\bar{2}$
$= (2,3,0,1,4)(\bar{2},\bar{3},\bar{0},\bar{1},\bar{4})\underline{4\bar{3}}0\bar{4}3\bar{2} = (2,3,0,1,4)(\bar{2},\bar{3},\bar{0},\bar{1},\bar{4})(2,0,1)(\bar{2},\bar{0},\bar{1})3\bar{4}0\bar{4}3\bar{2}$
$= (2,3,1,4,0)(\bar{2},\bar{3},\bar{1},\bar{4},\bar{0})3\underline{\bar{4}0}\bar{4}3\bar{2} = (2,3,1,4,0)(\bar{2},\bar{3},\bar{1},\bar{4},\bar{0})3(2,3,1)(\bar{2},\bar{3},\bar{1})\bar{0}4\bar{4}3\bar{2}$
$= (2,1,4,0,3)(\bar{2},\bar{1},\bar{4},\bar{0},\bar{3})1\bar{0}3\bar{2}.$ □

Since, $N^{(0\bar{1}\bar{2}\bar{3})} \geq N^{0\bar{1}\bar{2}\bar{3}} =< e >$, and
$N(t_0\bar{t_1}\bar{t_2}\bar{t_3})^{\{(1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3})\}} = 1\bar{0}\bar{3}\bar{2} \Rightarrow (1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3}) \in N^{(0\bar{1}\bar{2}\bar{3})}$
$\Rightarrow N^{(0\bar{1}\bar{2}\bar{3})} =< (1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3}) >= \{e, (1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3})\}$.

Therefore, the number of single cosets in the double coset $[0\bar{1}\bar{2}\bar{3}]$ are at most

$\frac{|N|}{|N^{(0123)}|} = \frac{5!}{2} = \frac{120}{2} = 60$. Each single coset has two different names.

By conjugating $(0\bar{1}\bar{2}\bar{3} \sim 1\bar{0}\bar{3}\bar{2})$ with the elements of transversal of $N^{(0\bar{1}\bar{2}\bar{3})}$, we obtain the following sixty single cosets of double coset $[0\bar{1}\bar{2}\bar{3}]$:
$0\bar{1}\bar{2}\bar{3} \sim 1\bar{0}\bar{3}\bar{2}, 1\bar{2}\bar{3}\bar{4} \sim 2\bar{1}\bar{4}\bar{3}, 2\bar{3}\bar{4}\bar{0} \sim 3\bar{2}\bar{0}\bar{4}, 3\bar{4}\bar{0}\bar{1} \sim 4\bar{3}\bar{1}\bar{0}, 1\bar{2}\bar{4}\bar{0} \sim 2\bar{1}\bar{0}\bar{4}, 4\bar{0}\bar{1}\bar{2} \sim 0\bar{4}\bar{2}\bar{1}$,
$2\bar{4}\bar{0}\bar{3} \sim 4\bar{2}\bar{3}\bar{0}, 2\bar{4}\bar{3}\bar{1} \sim 4\bar{2}\bar{1}\bar{3}, 2\bar{3}\bar{0}\bar{1} \sim 3\bar{2}\bar{1}\bar{0}, 4\bar{0}\bar{3}\bar{1} \sim 0\bar{4}\bar{1}\bar{3}, 3\bar{0}\bar{1}\bar{4} \sim 0\bar{3}\bar{4}\bar{1}, 4\bar{3}\bar{0}\bar{2} \sim 3\bar{4}\bar{2}\bar{0}$,

$3\bar{0}\bar{4}\bar{2} \sim 0\bar{3}\bar{2}\bar{4}, 3\bar{4}\bar{1}\bar{2} \sim 4\bar{3}\bar{2}\bar{1}, 3\bar{0}\bar{2}\bar{1} \sim 0\bar{3}\bar{1}\bar{2}, 1\bar{3}\bar{2}\bar{0} \sim 3\bar{1}\bar{0}\bar{2}, 0\bar{1}\bar{4}\bar{2} \sim 1\bar{0}\bar{2}\bar{4}, 4\bar{1}\bar{2}\bar{0} \sim 1\bar{4}\bar{0}\bar{2},$
$1\bar{4}\bar{3}\bar{0} \sim 4\bar{1}\bar{0}\bar{3}, 4\bar{0}\bar{2}\bar{3} \sim 0\bar{4}\bar{3}\bar{2}, 3\bar{1}\bar{2}\bar{4} \sim 1\bar{3}\bar{4}\bar{2}, 1\bar{4}\bar{2}\bar{3} \sim 4\bar{1}\bar{3}\bar{2}, 0\bar{2}\bar{1}\bar{4} \sim 2\bar{0}\bar{4}\bar{1}, 1\bar{2}\bar{0}\bar{3} \sim 2\bar{1}\bar{3}\bar{0},$
$0\bar{2}\bar{3}\bar{1} \sim 2\bar{0}\bar{1}\bar{3}, 0\bar{1}\bar{3}\bar{4} \sim 1\bar{0}\bar{4}\bar{3}, 0\bar{2}\bar{4}\bar{3} \sim 2\bar{0}\bar{3}\bar{4}, 3\bar{2}\bar{4}\bar{1} \sim 2\bar{3}\bar{1}\bar{4}, 1\bar{3}\bar{0}\bar{4} \sim 3\bar{1}\bar{4}\bar{0}, 2\bar{4}\bar{1}\bar{0} \sim 4\bar{2}\bar{0}\bar{1},$
$\bar{0}124 \sim \bar{1}042, \bar{1}243 \sim \bar{2}134, \bar{1}230 \sim \bar{2}103, \bar{2}430 \sim \bar{4}203, \bar{2}304 \sim \bar{3}240, \bar{2}341 \sim \bar{3}214,$
$\bar{4}301 \sim \bar{3}410, \bar{3}041 \sim \bar{0}314, \bar{3}402 \sim \bar{4}320, \bar{3}012 \sim \bar{0}321, \bar{4}021 \sim \bar{0}412, \bar{4}132 \sim \bar{1}420,$
$\bar{4}013 \sim \bar{0}413, \bar{2}401 \sim \bar{4}210, \bar{4}123 \sim \bar{1}432, \bar{0}132 \sim \bar{1}023, \bar{0}213 \sim \bar{2}031, \bar{4}032 \sim \bar{0}423,$
$\bar{4}312 \sim \bar{3}421, \bar{0}234 \sim \bar{2}043, \bar{1}324 \sim \bar{3}142, \bar{3}120 \sim \bar{1}302, \bar{1}340 \sim \bar{3}104, \bar{0}143 \sim \bar{1}034,$
$\bar{1}204 \sim \bar{2}140, \bar{0}241 \sim \bar{2}014, \bar{4}231 \sim \bar{2}413, \bar{4}130 \sim \bar{1}403, \bar{2}310 \sim \bar{3}201, \bar{3}024 \sim \bar{0}342.$

The orbits of $N^{(0\bar{1}\bar{2}\bar{3})}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{4\}$, $\{\bar{4}\}$, $\{2,3\}$, $\{\bar{2},\bar{3}\}$,$\{1,0\}$ and $\{\bar{1},\bar{0}\}$. Now, take one representative from each orbit, namely 4, $\bar{4}$, 3, $\bar{3}$, 0 and $\bar{0}$, then find the double cosets to which $Nt_0\bar{t}_1\bar{t}_2\bar{t}_3t_4$, $Nt_0\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4$, $Nt_0\bar{t}_1\bar{t}_2\bar{t}_3t_3$, $Nt_0\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_3$, $Nt_0\bar{t}_1\bar{t}_2\bar{t}_3t_0$ and $Nt_0\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_0$ belong. However,

$Nt_0\bar{t}_1\bar{t}_2\bar{t}_3t_4 = Nt_0\bar{t}_1\bar{t}_2\bar{t}_3t_4 \in Nt_0\bar{t}_1\bar{t}_2\bar{t}_3t_4N$ (New double coset denoted by $[0\bar{1}\bar{2}\bar{3}4]$).
$Nt_0\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4 = Nt_0\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4 \in [0\bar{1}\bar{2}\bar{3}]$ **(By Lemma-44).**
$Nt_0\bar{t}_1\bar{t}_2\bar{t}_3t_3 = Nt_0\bar{t}_1\bar{t}_2 \in [0\bar{1}\bar{2}]$.
$Nt_0\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_3 = Nt_0\bar{t}_1\bar{t}_2t_3 \in [0\bar{1}\bar{2}3]$.
$Nt_0\bar{t}_1\bar{t}_2\bar{t}_3t_0 = Nt_0\bar{t}_1\bar{t}_2\bar{t}_3t_0 \in [01210]$ **(By Lemma-45).**
$Nt_0\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_0 = Nt_0\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_0 \in [01023]$ **(By Lemma-46).**

Consider the double coset [01023]

$N^{(01023)} = N^{01023} =< e >$
Therefore, the number of single cosets in the double coset [01023] are at most

$$\frac{|N|}{|N^{(01023)}|}=\frac{5!}{1} = \frac{120}{1} = 120. \text{ Each single coset has only one name.}$$

$\Rightarrow T \cong S_5$.
Now, conjugating (01023) with elements of transversal of $N^{(01023)}$, we obtain the 120 single cosets of double coset [01023]:
01023, 31324, 41420, 41403, 31302, 21204, 21243, 31340, 01042, 01034, 41432, 21230,
12134, 42430, 02031, 02014, 42413, 32310, 32304, 42401, 12103, 12140, 02043, 32341,
40412, 20213, 30314, 30342, 20241, 10143, 10132, 20234, 40431, 40423, 30321, 10124,

$23240, 03041, 13142, 13120, 03024, 43421, 43410, 03012, 23214, 23201, 13104, 43402,$
$34301, 14102, 24203, 24231, 14130, 04032, 04021, 14123, 34320, 34312, 24210, 04013,$
$\bar0\bar1\bar0\bar2\bar4, \bar4\bar1\bar4\bar2\bar3, \bar3\bar1\bar3\bar2\bar0, \bar3\bar1\bar3\bar0\bar4, \bar4\bar1\bar4\bar0\bar2, \bar2\bar1\bar2\bar0\bar3, \bar2\bar1\bar2\bar3\bar4, \bar4\bar1\bar4\bar3\bar0, \bar0\bar1\bar0\bar3\bar2, \bar0\bar1\bar0\bar4\bar3, \bar3\bar1\bar3\bar4\bar2, \bar2\bar1\bar2\bar4\bar0,$
$\bar1\bar2\bar1\bar4\bar3, \bar3\bar2\bar3\bar4\bar0, \bar0\bar2\bar0\bar4\bar1, \bar0\bar2\bar0\bar1\bar3, \bar3\bar2\bar3\bar1\bar4, \bar4\bar2\bar4\bar1\bar0, \bar4\bar2\bar4\bar0\bar3, \bar3\bar2\bar3\bar0\bar1, \bar1\bar2\bar1\bar0\bar4, \bar1\bar2\bar1\bar3\bar0, \bar0\bar2\bar0\bar3\bar4, \bar4\bar2\bar4\bar3\bar1,$
$\bar4\bar0\bar4\bar1\bar3, \bar3\bar0\bar3\bar1\bar2, \bar2\bar0\bar2\bar1\bar4, \bar2\bar0\bar2\bar4\bar3, \bar3\bar0\bar3\bar4\bar1, \bar1\bar0\bar1\bar4\bar2, \bar1\bar0\bar1\bar2\bar3, \bar3\bar0\bar3\bar2\bar4, \bar4\bar0\bar4\bar2\bar1, \bar4\bar0\bar4\bar3\bar2, \bar2\bar0\bar2\bar3\bar1, \bar1\bar0\bar1\bar3\bar4,$
$\bar2\bar4\bar2\bar3\bar0, \bar0\bar4\bar0\bar3\bar1, \bar1\bar4\bar1\bar3\bar2, \bar1\bar4\bar1\bar2\bar0, \bar0\bar4\bar0\bar2\bar3, \bar3\bar4\bar3\bar2\bar1, \bar3\bar4\bar3\bar1\bar0, \bar0\bar4\bar0\bar1\bar2, \bar2\bar4\bar2\bar1\bar3, \bar2\bar4\bar2\bar0\bar1, \bar1\bar4\bar1\bar0\bar3, \bar3\bar4\bar3\bar0\bar2,$
$\bar2\bar3\bar2\bar0\bar4, \bar4\bar3\bar4\bar0\bar1, \bar1\bar3\bar1\bar0\bar2, \bar1\bar3\bar1\bar2\bar4, \bar4\bar3\bar4\bar2\bar0, \bar0\bar3\bar0\bar2\bar1, \bar0\bar3\bar0\bar1\bar4, \bar4\bar3\bar4\bar1\bar2, \bar2\bar3\bar2\bar1\bar0, \bar2\bar3\bar2\bar4\bar1, \bar1\bar3\bar1\bar4\bar0, \bar0\bar3\bar0\bar4\bar2$

The orbits of $N^{(01023)}$ on $\{0,1,2,3,4,\bar0,\bar1,\bar2,\bar3,\bar4\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{\bar0\}$, $\{\bar1\}$, $\{\bar2\}$, $\{\bar3\}$ and $\{\bar4\}$. Now, take one representative from each orbit and find the double cosets to which $Nt_0t_1t_0t_2t_3t_0$, $Nt_0t_1t_0t_2t_3t_1$, $Nt_0t_1t_0t_2t_3t_2$, $Nt_0t_1t_0t_2t_3t_3$, $Nt_0t_1t_0t_2t_3t_4$, $Nt_0t_1t_0t_2t_3t_{\bar0}$, $Nt_0t_1t_0t_2t_3t_{\bar1}$, $Nt_0t_1t_0t_2t_3t_{\bar2}$, $Nt_0t_1t_0t_2t_3t_{\bar3}$, $Nt_0t_1t_0t_2t_3t_{\bar4}$ belong. However,

$Nt_0t_1t_0t_2t_3t_0 = Nt_0t_1t_0t_2t_3t_0 \in [012\bar3\bar0]$**(By Lemma-47).**
$Nt_0t_1t_0t_2t_3t_1 = Nt_0t_1t_0t_2t_3t_1 \in [01023]$ **(By Lemma-48).**
$Nt_0t_1t_0t_2t_3t_2 = Nt_0t_1t_0t_2t_3t_2 \in Nt_0t_1t_0t_2t_3t_2N$ (New double coset denote by [010232]).
$Nt_0t_1t_0t_2t_3t_3 = Nt_0t_1t_0t_2t_{\bar3} \in [0\bar1\bar23]$ **(By Lemma-49).**
$Nt_0t_1t_0t_2t_3t_4 = Nt_0t_1t_0t_2t_3t_4 \in [0\bar1\bar2\bar3]$ **(By Lemma-50).**
$Nt_0t_1t_0t_2t_3t_{\bar0} = Nt_0t_1t_0t_2t_3t_{\bar0} \in [01203]$ **(By Lemma-51).**
$Nt_0t_1t_0t_2t_3t_{\bar1} = Nt_0t_1t_0t_2t_3t_{\bar1} \in [01201]$ **(By Lemma-52).**
$Nt_0t_1t_0t_2t_3t_{\bar2} = Nt_0t_1t_0t_2t_3t_{\bar2} \in [01023]$ **(By Lemma-53).**
$Nt_0t_1t_0t_2t_3t_{\bar3} = Nt_0t_1t_0t_2 \in [0102]$.
$Nt_0t_1t_0t_2t_3t_{\bar4} = Nt_0t_1t_0t_2t_3t_{\bar4} \in [01214]$ **(By Lemma-54).**

Consider the double coset [01201]
Claim: $01201 \sim \bar0\bar4\bar2\bar0\bar4$

Proof. By our relation-I, $(2,3,4)(\bar2,\bar3,\bar4)\bar10 = \bar01$ conjugating with element of N, we obtain the following relations:

- $((2,3,4)(\bar2,\bar3,\bar4)\bar10)^{\{(1,\bar1)(2,\bar2)(3,\bar4)(4,\bar3)(0,\bar0)\}} = (\bar01)^{\{(1,\bar1)(2,\bar2)(3,\bar4)(4,\bar3)(0,\bar0)\}}$
  $\Rightarrow (2,4,3)(\bar2,\bar4,\bar3)1\bar0 = 0\bar1$
- $((2,3,4)(\bar2,\bar3,\bar4)\bar10)^{\{(2,0,4)(\bar2,\bar0,\bar4)\}} = (\bar01)^{\{(2,0,4)(\bar2,\bar0,\bar4)\}}$

$\Rightarrow (2,0,3)(\bar{2},\bar{0},\bar{3})\bar{1}4 = \bar{4}1$

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{0},4,\bar{2})(2,\bar{1},0,\bar{4})(3,\bar{3})\}} = (\bar{0}1)^{\{(1,\bar{0},4,\bar{2})(2,\bar{1},0,\bar{4})(3,\bar{3})\}}$
  $\Rightarrow (2,1,3)(\bar{2},\bar{1},\bar{3})0\bar{4} = 4\bar{0}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{3})(4,\bar{0})(0,\bar{4})\}} = (\bar{0}1)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{3})(4,\bar{0})(0,\bar{4})\}}$
  $\Rightarrow (2,3,0)(\bar{2},\bar{3},\bar{0})1\bar{4} = 4\bar{1}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,3)(4,0)(1,\bar{3})(4,\bar{0})\}} = (\bar{0}1)^{\{(1,3)(4,0)(1,\bar{3})(4,\bar{0})\}}$
  $\Rightarrow (2,1,0)(\bar{2},\bar{1},\bar{0})\bar{3}4 = \bar{4}3$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,0,4)(\bar{1},\bar{0},\bar{4})\}} = (\bar{0}1)^{\{(1,0,4)(\bar{1},\bar{0},\bar{4})\}}$
  $\Rightarrow (2,3,1)(\bar{2},\bar{3},\bar{1})\bar{0}4 = \bar{4}0$

And, by our relation-II, $(0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})2104 = \bar{0}\bar{1}\bar{2}\bar{3}$ conjugating with element of N, we obtain the following relations:

- $((0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})2104)^{\{(1,4)(3,0)(\bar{1},\bar{4})(\bar{3},\bar{0})\}} = (\bar{0}\bar{1}\bar{2}\bar{3})^{\{(1,4)(3,0)(\bar{1},\bar{4})(\bar{3},\bar{0})\}}$
  $\Rightarrow (3,0,4,1,2)(\bar{3},\bar{0},\bar{4},\bar{1},\bar{2})2431 = \bar{3}\bar{4}\bar{2}\bar{0}$
- $((0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})2104)^{\{(1,4,2)(\bar{1},\bar{4},\bar{2})\}} = (\bar{0}\bar{1}\bar{2}\bar{3})^{\{(1,4,2)(\bar{1},\bar{4},\bar{2})\}}$
  $\Rightarrow (0,3,4,2,1)(\bar{0},\bar{3},\bar{4},\bar{2},\bar{1})1402 = \bar{0}\bar{4}\bar{1}\bar{3}$

From the relations above, we have
$01201 = 012\underline{0\bar{1}}\bar{1} = 012(2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}\bar{1} = (2,4,3)(\bar{2},\bar{4},\bar{3})0141\bar{0}\bar{1}$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})01\bar{4}\underline{\bar{4}1}\bar{0}\bar{1} = (2,4,3)(\bar{2},\bar{4},\bar{3})01\bar{4}(2,0,3)(\bar{2},\bar{0},\bar{3})\bar{1}4\bar{0}\bar{1}$
$= (2,4)(3,0)(\bar{2},\bar{4})(\bar{3},\bar{0})31\bar{4}\bar{1}\underline{4\bar{0}}\bar{1} = (2,4)(3,0)(\bar{2},\bar{4})(\bar{3},\bar{0})31\bar{4}\bar{1}(2,1,3)(\bar{2},\bar{1},\bar{3})0\bar{4}\bar{1}$
$= (2,4,1,3,0)(\bar{2},\bar{4},\bar{1},\bar{3},\bar{0})23\bar{4}\bar{3}0\bar{4}\bar{1} = (2,4,1,3,0)(\bar{2},\bar{4},\bar{1},\bar{3},\bar{0})23\bar{4}\bar{3}04\underline{4\bar{1}}$
$= (2,4,1,3,0)(\bar{2},\bar{4},\bar{1},\bar{3},\bar{0})23\bar{4}\bar{3}04(2,3,0)(\bar{2},\bar{3},\bar{0})1\bar{4} = (2,4,1,0,3)(\bar{2},\bar{4},\bar{1},\bar{0},\bar{3})3\underline{0\bar{4}}\bar{0}241\bar{4}$
$= (2,4,1,0,3)(\bar{2},\bar{4},\bar{1},\bar{0},\bar{3})3(1,2,3)(\bar{1},\bar{2},\bar{3})4\bar{0}\bar{0}241\bar{4} = (2,4,3,1,0)(\bar{2},\bar{4},\bar{3},\bar{1},\bar{0})\underline{1402}41\bar{4}$
$= (2,4,3,1,0)(\bar{2},\bar{4},\bar{3},\bar{1},\bar{0})(0,1,2,4,3)(\bar{0},\bar{1},\bar{2},\bar{4},\bar{3})\bar{0}\bar{4}\bar{1}\bar{3}41\bar{4}$
$= (2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0})\bar{0}\bar{4}\bar{1}\bar{3}41\bar{4} = (2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0})\bar{0}4\underline{4\bar{1}}\bar{3}41\bar{4}$
$= (2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0})\bar{0}4(2,3,0)(\bar{2},\bar{3},\bar{0})1\bar{4}\bar{3}41\bar{4} = (2,0,4)(\bar{2},\bar{0},\bar{4})\bar{2}41\bar{4}\underline{\bar{3}4}1\bar{4}$
$= (2,0,4)(\bar{2},\bar{0},\bar{4})\bar{2}41\bar{4}(2,0,1)(\bar{2},\bar{0},\bar{1})\bar{4}31\bar{4} = (2,1)(4,0)(\bar{2},\bar{1})(\bar{4},\bar{0})\bar{0}42\bar{4}\bar{4}31\bar{4}$
$= (2,1)(4,0)(\bar{2},\bar{1})(\bar{4},\bar{0})\underline{\bar{0}4}2431\bar{4} = (2,1)(4,0)(\bar{2},\bar{1})(\bar{4},\bar{0})(2,1,3)(\bar{2},\bar{1},\bar{3})\bar{4}02431\bar{4}$
$= (2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0})\bar{4}0\underline{2431}\bar{4} = (2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0})\bar{4}0(3,2,1,4,0)(\bar{3},\bar{2},\bar{1},\bar{4},\bar{0})\bar{3}\bar{4}\bar{2}\bar{0}\bar{4}$
$= (1,4,3)(\bar{1},\bar{4},\bar{3})\bar{0}3\bar{3}\bar{4}\bar{2}\bar{0}\bar{4} = (1,4,3)(\bar{1},\bar{4},\bar{3})\bar{0}\bar{4}\bar{2}\bar{0}\bar{4}.$ □

Since, $N^{(01201)} \geq N^{01201} =< e >$, and
$N(t_0t_1t_2t_0t_1)^{\{(1,\bar{4})(2,\bar{2})(3,\bar{3})(4,\bar{1})(0,\bar{0})\}} = \overline{04204} \Rightarrow (1,\bar{4})(2,\bar{2})(3,\bar{3})(4,\bar{1})(0,\bar{0}) \in N^{(01201)}$
$\Rightarrow N^{(01201)} =< (1,\bar{4})(2,\bar{2})(3,\bar{3})(4,\bar{1})(0,\bar{0}) >= \{e, (1,\bar{4})(2,\bar{2})(3,\bar{3})(4,\bar{1})(0,\bar{0})\}$.

Therefore, the number of single cosets in the double coset [01201] are at most

$$\frac{|N|}{|N^{(01201)}|} = \frac{5!}{2} = \frac{120}{2} = 60.$$ Each single coset has two different names.

By conjugating ($01201 \sim \overline{04204}$) with the elements of transversal of $N^{(01201)}$, we obtain the sixty single cosets of double coset [01201]:
$01201 \sim \overline{04204}, 12312 \sim \overline{10310}, 03203 \sim \overline{01201}, 23423 \sim \overline{21421}, 10410 \sim \overline{12412},$
$14314 \sim \overline{12312}, 34034 \sim \overline{32032}, 21321 \sim \overline{24324}, 21021 \sim \overline{23023}, 20420 \sim \overline{23423},$
$12412 \sim \overline{13413}, 40140 \sim \overline{43143}, 42042 \sim \overline{43043}, 32432 \sim \overline{30430}, 32132 \sim \overline{34134},$
$24024 \sim \overline{21021}, 31031 \sim \overline{34034}, 24324 \sim \overline{20320}, 23023 \sim \overline{24024}, 34134 \sim \overline{30130},$
$03103 \sim \overline{04104}, 43043 \sim \overline{41041}, 40340 \sim \overline{42342}, 43243 \sim \overline{40240}, 43143 \sim \overline{42142},$
$30130 \sim \overline{32132}, 42142 \sim \overline{40140}, 30430 \sim \overline{31431}, 40240 \sim \overline{41241}, 14214 \sim \overline{10210},$
$04104 \sim \overline{02102}, 01401 \sim \overline{03403}, 04304 \sim \overline{01301}, 30230 \sim \overline{34234}, 04204 \sim \overline{03203},$
$41241 \sim \overline{43243}, 41041 \sim \overline{42042}, 41341 \sim \overline{40340}, 01301 \sim \overline{02302}, 31231 \sim \overline{30230},$
$20320 \sim \overline{21321}, 10210 \sim \overline{13213}, 02102 \sim \overline{03103}, 12012 \sim \overline{14014}, 10310 \sim \overline{14314},$
$02302 \sim \overline{04304}, 34234 \sim \overline{31231}, 32032 \sim \overline{31031}, 02402 \sim \overline{01401}, 42342 \sim \overline{41341},$
$31431 \sim \overline{32432}, 21421 \sim \overline{20420}, 13213 \sim \overline{14214}, 23123 \sim \overline{20120}, 13013 \sim \overline{12012},$
$13413 \sim \overline{10410}, 03403 \sim \overline{02402}, 20120 \sim \overline{24124}, 24124 \sim \overline{23123}, 14014 \sim \overline{13013}.$

The orbits of $N^{(01201)}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{1,\bar{4}\}$, $\{2,\bar{2}\}$, $\{3,\bar{3}\}$, $\{4,\bar{1}\}$ and $\{0,\bar{0}\}$. Now, take one representative from each orbit, namely 1, 2, 3, $\bar{1}$ and $\bar{0}$, then find the double cosets to which $Nt_0t_1t_2t_0t_1t_1$, $Nt_0t_1t_2t_0t_1t_2$, $Nt_0t_1t_2t_0t_1t_3$, $Nt_0t_1t_2t_0t_1\bar{t_1}$ and $Nt_0t_1t_2t_0t_1t_0$ belong. However,

$Nt_0t_1t_2t_0t_1t_1 = Nt_0t_1t_2t_0\bar{t_1} \in [0120\bar{1}]$.
$Nt_0t_1t_2t_0t_1t_2 = Nt_0t_1t_2t_0t_1t_2 \in [01201]$ **(By Lemma-55).**
$Nt_0t_1t_2t_0t_1t_3 = Nt_0t_1t_2t_0t_1t_3 \in [01201]$ **(By Lemma-56).**
$Nt_0t_1t_2t_0t_1\bar{t_1} = Nt_0t_1t_2t_0 \in [0120]$
$Nt_0t_1t_2t_0t_1t_0 = Nt_0t_1t_2t_0t_1t_0 \in [01023]$ **(By Lemma-57).**

Consider the double coset [01204]

Claim: $01204 \sim 42140$

Proof. By our relation-I, $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtain the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{2})(2,\bar{1})(3,\bar{3})(4,\bar{4})(0,\bar{0})\}} = (\bar{0}1)^{\{(1,\bar{2})(2,\bar{1})(3,\bar{3})(4,\bar{4})(0,\bar{0})\}}$
  $\Rightarrow (1,3,4)(\bar{1},\bar{3},\bar{4})2\bar{0} = 0\bar{2}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{3},0,\bar{4})(2,\bar{2})(3,\bar{0},4,\bar{1})\}} = (\bar{0}1)^{\{(1,\bar{3},0,\bar{4})(2,\bar{2})(3,\bar{0},4,\bar{1})\}}$
  $\Rightarrow (2,0,1)(\bar{2},\bar{0},\bar{1})3\bar{4} = 4\bar{3}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,0,4)(\bar{1},\bar{0},\bar{4})\}} = (\bar{0}1)^{\{(1,0,4)(\bar{1},\bar{0},\bar{4})\}}$
  $\Rightarrow (2,3,1)(\bar{2},\bar{3},\bar{1})\bar{0}4 = \bar{4}0$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(2,0,4)(\bar{2},\bar{0},\bar{4})\}} = (\bar{0}1)^{\{(2,0,4)(\bar{2},\bar{0},\bar{4})\}}$
  $\Rightarrow (0,3,2)(\bar{0},\bar{3},\bar{2})\bar{1}4 = \bar{4}1$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{3},4,\bar{0})(2,\bar{2})(3,\bar{4},0,\bar{1})\}} = (\bar{0}1)^{\{(1,\bar{3},4,\bar{0})(2,\bar{2})(3,\bar{4},0,\bar{1})\}}$
  $\Rightarrow (2,4,0)(\bar{2},\bar{4},\bar{0})3\bar{1} = 1\bar{3}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{4},0,\bar{2})(2,\bar{1},4,\bar{0})(3,\bar{3})\}} = (\bar{0}1)^{\{(1,\bar{4},0,\bar{2})(2,\bar{1},4,\bar{0})(3,\bar{3})\}}$
  $\Rightarrow (1,3,0)(\bar{1},\bar{3},\bar{0})4\bar{2} = 2\bar{4}$

From the relations above, we have

$01204 = 012\bar{0}\underline{\bar{0}4} = 012\bar{0}(2,1,3)(\bar{2},\bar{1},\bar{3})\bar{4}0 = (2,1,3)(\bar{2},\bar{1},\bar{3})031\bar{0}\bar{4}0$
$= (2,1,3)(\bar{2},\bar{1},\bar{3})031\underline{\bar{0}4}\bar{4}40 = (2,1,3)(\bar{2},\bar{1},\bar{3})031(2,1,3)(\bar{2},\bar{1},\bar{3})\bar{4}040$
$= (2,3,1)(\bar{2},\bar{3},\bar{1})02\underline{3\bar{4}}040 = (2,3,1)(\bar{2},\bar{3},\bar{1})02(2,0,1)(\bar{2},\bar{1},\bar{0})4\bar{3}040$
$= (2,3,0)(\bar{2},\bar{3},\bar{0})214\bar{3}040 = (2,3,0)(\bar{2},\bar{3},\bar{0})2\bar{1}\underline{\bar{1}4}\bar{3}040$
$= (2,3,0)(\bar{2},\bar{3},\bar{0})2\bar{1}(0,2,3)(\bar{0},\bar{2},\bar{3})\bar{4}1\bar{3}040 = (2,0,3)(\bar{2},\bar{0},\bar{3})3\bar{1}\bar{4}\underline{1\bar{3}}040$
$= (2,0,3)(\bar{2},\bar{0},\bar{3})3\bar{1}\bar{4}(2,4,0)(\bar{2},\bar{4},\bar{0})3\bar{1}040 = (0,3,4)(\bar{0},\bar{3},\bar{4})3\bar{1}\bar{0}3\underline{\bar{1}0}40$
$= (0,3,4)(\bar{0},\bar{3},\bar{4})3\bar{1}\bar{0}3(2,4,3)(\bar{2},\bar{4},\bar{3})\bar{0}140 = (0,2,4)(\bar{0},\bar{2},\bar{4})2\bar{1}\bar{0}\underline{2\bar{0}}140$
$= (0,2,4)(\bar{0},\bar{2},\bar{4})2\bar{1}\bar{0}(1,4,3)(\bar{1},\bar{4},\bar{3})0\bar{2}140 = (0,2,3,1,4)(\bar{0},\bar{2},\bar{3},\bar{1},\bar{4})2\bar{4}\bar{0}0\bar{2}140$
$= (0,2,3,1,4)(\bar{0},\bar{2},\bar{3},\bar{1},\bar{4})\underline{2\bar{4}}\bar{2}140 = (0,2,3,1,4)(\bar{0},\bar{2},\bar{3},\bar{1},\bar{4})(1,3,0)(\bar{1},\bar{3},\bar{0})4\bar{2}\bar{2}140$
$= (0,2)(1,4)(\bar{0},\bar{2})(\bar{1},\bar{4})42140.$ ∎

Since, $N^{(01204)} \geq N^{01204} =< e >$, and

$N(t_0t_1t_2t_0t_4)^{\{(1,2)(4,0)(\bar{1},\bar{2})(\bar{4},\bar{0})\}} = 42140 \Rightarrow (1,2)(4,0)(\bar{1},\bar{2})(\bar{4},\bar{0}) \in N^{(01204)}$

$\Rightarrow N^{(01204)} =< (1,2)(4,0)(\bar{1},\bar{2})(\bar{4},\bar{0}) >= \{e, (1,2)(4,0)(\bar{1},\bar{2})(\bar{4},\bar{0})\}$.

Therefore, the number of single cosets in the double coset [01204] are at most

$$\frac{|N|}{|N^{(01204)}|} = \frac{5!}{2} = \frac{120}{2} = 60.$$ Each single coset has two different names.

By conjugating $(01204 \sim 42140)$ with the elements of transversal of $N^{(01204)}$, we obtain the following sixty single cosets of double coset [01204]:

$01204 \sim 42140, 12310 \sim 03201, 23421 \sim 14312, 34032 \sim 20423, 12413 \sim 34231,$
$40143 \sim 31034, 24021 \sim 10412, 24320 \sim 03402, 23024 \sim 40342, 43142 \sim 21324,$
$30132 \sim 21023, 43041 \sim 10314, 30431 \sim 14013, 01403 \sim 34130, 03104 \sim 41340,$
$30234 \sim 42043, 32430 \sim 04203, 41243 \sim 32134, 04102 \sim 21420, 20124 \sim 41042,$
$12014 \sim 40241, 31230 \sim 02103, 32031 \sim 10213, 14210 \sim 02401, 02304 \sim 43240,$
$01302 \sim 23120, 13214 \sim 42341, 20321 \sim 13012, 13410 \sim 04301, 24123 \sim 31432,$
$\overline{01203} \sim \overline{32130}, \overline{12410} \sim \overline{04201}, \overline{12314} \sim \overline{43241}, \overline{24321} \sim \overline{13412}, \overline{23021} \sim \overline{10312},$
$\overline{23420} \sim \overline{04302}, \overline{43042} \sim \overline{20324}, \overline{24023} \sim \overline{30432}, \overline{34132} \sim \overline{21423}, \overline{34031} \sim \overline{10413},$
$\overline{30134} \sim \overline{41043}, \overline{04103} \sim \overline{31430}, \overline{40243} \sim \overline{32034}, \overline{40142} \sim \overline{21024}, \overline{41240} \sim \overline{02104},$
$\overline{10214} \sim \overline{42041}, \overline{01304} \sim \overline{43140}, \overline{40341} \sim \overline{13014}, \overline{13210} \sim \overline{02301}, \overline{21320} \sim \overline{03102},$
$\overline{42340} \sim \overline{03204}, \overline{31234} \sim \overline{42143}, \overline{20123} \sim \overline{31032}, \overline{01402} \sim \overline{24120}, \overline{12013} \sim \overline{30231},$
$\overline{32431} \sim \overline{14213}, \overline{14310} \sim \overline{03401}, \overline{02403} \sim \overline{34230}, \overline{23124} \sim \overline{41342}, \overline{20421} \sim \overline{14012}.$

The orbits of $N^{(01204)}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{3\}$, $\{\bar{3}\}$, $\{1,2\}$, $\{\bar{1},\bar{2}\}$,$\{4,0\}$ and $\{\bar{4},\bar{0}\}$. Now, take one representative from each orbit, namely 3, $\bar{3}$, 1, $\bar{1}$, 4 and $\bar{4}$, then find the double cosets to which $Nt_0t_1t_2t_0t_4t_3$, $Nt_0t_1t_2t_0t_4\bar{t_3}$, $Nt_0t_1t_2t_0t_4t_1$, $Nt_0t_1t_2t_0t_4\bar{t_1}$, $Nt_0t_1t_2t_0t_4t_4$ and $Nt_0t_1t_2t_0t_4\bar{t_4}$ belong. However,

$Nt_0t_1t_2t_0t_4t_3 = Nt_0t_1t_2t_0t_4t_3 \in [0121]$ **(By Lemma-58)**.
$Nt_0t_1t_2t_0t_4\bar{t_3} = Nt_0t_1t_2t_0t_4\bar{t_3} \in [01210]$ **(By Lemma-59)**.
$Nt_0t_1t_2t_0t_4t_1 = Nt_0t_1t_2t_0t_4t_1 \in [0123]$ **(By Lemma-60)**.
$Nt_0t_1t_2t_0t_4\bar{t_1} = Nt_0t_1t_2t_0t_4\bar{t_1} \in [01203]$ **(By Lemma-61)**.
$Nt_0t_1t_2t_0t_4t_4 = Nt_0t_1t_2t_0\bar{t_4} \in [0120\bar{4}]$.
$Nt_0t_1t_2t_0t_4\bar{t_4} = Nt_0t_1t_2t_0 \in [0120]$.

Consider the double coset $[0120\bar{3}]$

$N^{(0120\bar{3})} = N^{0120\bar{3}} =< e >$

Therefore, the number of single cosets in the double coset $[0120\bar{3}]$ are at most

$\frac{|N|}{|N^{(0120\bar{3})}|} = \frac{5!}{1} = \frac{120}{1} = 120$. Each single coset has only one name.

$\Rightarrow T \cong S_5$.

Now, conjugating $(0120\bar{3})$ with elements of transversal of $N^{(0120\bar{3})}$, we obtain the 120 single cosets of double coset $[0120\bar{3}]$:

$1020\bar{3}, 3123\bar{4}, 4124\bar{0}, 4104\bar{3}, 3103\bar{2}, 2102\bar{4}, 2142\bar{3}, 3143\bar{0}, 0140\bar{2}, 0130\bar{4}, 4134\bar{2}, 2132\bar{0},$
$1231\bar{4}, 4234\bar{0}, 0230\bar{1}, 0210\bar{4}, 4214\bar{3}, 3213\bar{0}, 3203\bar{4}, 4204\bar{1}, 1201\bar{3}, 1241\bar{0}, 0240\bar{3}, 3243\bar{1},$
$4014\bar{2}, 2012\bar{3}, 3013\bar{4}, 3043\bar{2}, 2042\bar{1}, 1041\bar{3}, 1031\bar{2}, 2032\bar{4}, 4034\bar{1}, 4024\bar{3}, 3023\bar{1}, 1021\bar{4},$
$2342\bar{0}, 0340\bar{1}, 1341\bar{2}, 1321\bar{0}, 0320\bar{4}, 4324\bar{1}, 4314\bar{0}, 0310\bar{2}, 2312\bar{4}, 2302\bar{1}, 1301\bar{4}, 4304\bar{2},$
$3403\bar{1}, 1401\bar{2}, 2402\bar{3}, 2432\bar{1}, 1431\bar{0}, 0430\bar{2}, 0420\bar{1}, 1421\bar{3}, 3423\bar{0}, 3413\bar{2}, 2412\bar{0}, 0410\bar{3},$
$\bar{0}\bar{1}\bar{2}\bar{0}4, \bar{4}\bar{1}\bar{2}\bar{4}3, \bar{3}\bar{1}\bar{2}\bar{3}0, \bar{3}\bar{1}\bar{0}\bar{3}4, \bar{4}\bar{1}\bar{0}\bar{4}2, \bar{2}\bar{1}\bar{0}\bar{2}3, \bar{2}\bar{1}\bar{3}\bar{2}4, \bar{4}\bar{1}\bar{3}\bar{4}0, \bar{0}\bar{1}\bar{3}\bar{0}2, \bar{0}\bar{1}\bar{4}\bar{0}3, \bar{3}\bar{1}\bar{4}\bar{3}2, \bar{2}\bar{1}\bar{4}\bar{2}0,$
$\bar{1}\bar{2}\bar{4}\bar{1}3, \bar{3}\bar{2}\bar{4}\bar{3}0, \bar{0}\bar{2}\bar{4}\bar{0}1, \bar{0}\bar{2}\bar{1}\bar{0}3, \bar{3}\bar{2}\bar{1}\bar{3}4, \bar{4}\bar{2}\bar{1}\bar{4}0, \bar{4}\bar{2}\bar{0}\bar{4}3, \bar{3}\bar{2}\bar{0}\bar{3}1, \bar{1}\bar{2}\bar{0}\bar{1}4, \bar{1}\bar{2}\bar{3}\bar{1}0, \bar{0}\bar{2}\bar{3}\bar{0}4, \bar{4}\bar{2}\bar{3}\bar{4}1,$
$\bar{4}\bar{0}\bar{1}\bar{4}3, \bar{3}\bar{0}\bar{1}\bar{3}2, \bar{2}\bar{0}\bar{1}\bar{2}4, \bar{2}\bar{0}\bar{4}\bar{2}3, \bar{3}\bar{0}\bar{4}\bar{3}1, \bar{1}\bar{0}\bar{4}\bar{1}2, \bar{1}\bar{0}\bar{2}\bar{1}3, \bar{3}\bar{0}\bar{2}\bar{3}4, \bar{4}\bar{0}\bar{2}\bar{4}1, \bar{4}\bar{0}\bar{3}\bar{4}2, \bar{2}\bar{0}\bar{3}\bar{2}1, \bar{1}\bar{0}\bar{3}\bar{1}4,$
$\bar{2}\bar{4}\bar{3}\bar{2}0, \bar{0}\bar{4}\bar{3}\bar{0}1, \bar{1}\bar{4}\bar{3}\bar{1}2, \bar{1}\bar{4}\bar{2}\bar{1}0, \bar{0}\bar{4}\bar{2}\bar{0}3, \bar{3}\bar{4}\bar{2}\bar{3}1, \bar{3}\bar{4}\bar{1}\bar{3}0, \bar{0}\bar{4}\bar{1}\bar{0}2, \bar{2}\bar{4}\bar{1}\bar{2}3, \bar{2}\bar{4}\bar{0}\bar{2}1, \bar{1}\bar{4}\bar{0}\bar{1}3, \bar{3}\bar{4}\bar{0}\bar{3}2,$
$\bar{2}\bar{3}\bar{0}\bar{2}4, \bar{4}\bar{3}\bar{0}\bar{4}1, \bar{1}\bar{3}\bar{0}\bar{1}2, \bar{1}\bar{3}\bar{2}\bar{1}4, \bar{4}\bar{3}\bar{2}\bar{4}0, \bar{0}\bar{3}\bar{2}\bar{0}1, \bar{0}\bar{3}\bar{1}\bar{0}4, \bar{4}\bar{3}\bar{1}\bar{4}2, \bar{2}\bar{3}\bar{1}\bar{2}0, \bar{2}\bar{3}\bar{4}\bar{2}1, \bar{1}\bar{3}\bar{4}\bar{1}0, \bar{0}\bar{3}\bar{4}\bar{0}2.$

The orbits of $N^{(0120\bar{3})}$ on $\{0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, $\{\bar{3}\}$ and $\{\bar{4}\}$. Now, take one representative from each orbit, and find the double cosets to which $Nt_0t_1t_2t_0\bar{t_3}t_0$, $Nt_0t_1t_2t_0\bar{t_3}t_1$, $Nt_0t_1t_2t_0\bar{t_3}t_2$, $Nt_0t_1t_2t_0\bar{t_3}t_3$, $Nt_0t_1t_2t_0\bar{t_3}t_4$, $Nt_0t_1t_2t_0\bar{t_3}\bar{t_0}$, $Nt_0t_1t_2t_0\bar{t_3}\bar{t_1}$, $Nt_0t_1t_2t_0\bar{t_3}\bar{t_2}$, $Nt_0t_1t_2t_0\bar{t_3}\bar{t_3}$ and $Nt_0t_1t_2t_0\bar{t_3}\bar{t_4}$ belong. However,

$Nt_0t_1t_2t_0\bar{t_3}t_0 = Nt_0t_1t_2t_0\bar{t_3}t_0 \in [0123]$ **(By Lemma-62)**.
$Nt_0t_1t_0t_2\bar{t_3}t_1 = Nt_0t_1t_2t_0\bar{t_3}t_1 \in [0120\bar{3}]$ **(By Lemma-63)**.
$Nt_0t_1t_0t_2\bar{t_3}t_2 = Nt_0t_1t_2t_0\bar{t_3}t_2 \in [0123]$ **(By Lemma-64)**.
$Nt_0t_1t_0t_2\bar{t_3}t_3 = Nt_0t_1t_2t_0 \in [0120]$.
$Nt_0t_1t_0t_2\bar{t_3}t_4 = Nt_0t_1t_2t_0\bar{t_3}t_4 \in [0120]$ **(By Lemma-65)**.
$Nt_0t_1t_0t_2\bar{t_3}\bar{t_0} = Nt_0t_1t_2t_0\bar{t_3}\bar{t_0} \in [01\bar{2}\bar{0}]$ **(By Lemma-66)**.

$Nt_0t_1t_0t_2\bar{t_3}\bar{t_1} = Nt_0t_1t_2t_0\bar{t_3}\bar{t_1} \in [0120\bar{3}]$ **(By Lemma-67).**
$Nt_0t_1t_0t_2\bar{t_3}\bar{t_2} = Nt_0t_1t_2t_0\bar{t_3}\bar{t_2} \in [012\bar{2}\bar{3}]$ **(By Lemma-68).**
$Nt_0t_1t_0t_2\bar{t_3}\bar{t_3} = Nt_0t_1t_2t_0t_3 \in [01203]$.
$Nt_0t_1t_0t_2\bar{t_3}\bar{t_4} = Nt_0t_1t_2t_0\bar{t_3}\bar{t_4} \in [01204]$ **(By Lemma-69).**

Consider the double coset [01214]
Claim: $01214 \sim 32124$

Proof. By our relation-I, $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtain the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,2,3,4,0)(\bar{1},\bar{2},\bar{3},\bar{4},\bar{0})\}} = (\bar{0}1)^{\{(1,2,3,4,0)(\bar{1},\bar{2},\bar{3},\bar{4},\bar{0})\}}$
  $\Rightarrow (3,4,0)(\bar{3},\bar{4},\bar{0})\bar{2}1 = \bar{1}2$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{1})(2,\bar{0})(3,\bar{3})(4,\bar{4})(0,\bar{2})\}} = (\bar{0}1)^{\{(1,\bar{1})(2,\bar{0})(3,\bar{3})(4,\bar{4})(0,\bar{2})\}}$
  $\Rightarrow (0,3,4)(\bar{0},\bar{3},\bar{4})1\bar{2} = 2\bar{1}$

From the relations above, we have
$01214 = 01\bar{2}\underline{\bar{2}1}4 = 01\bar{2}(3,0,4)(\bar{3},\bar{0},\bar{4})\bar{1}24 = (3,0,4)(\bar{3},\bar{0},\bar{4})4\underline{1\bar{2}}\bar{1}24$
$= (3,0,4)(\bar{3},\bar{0},\bar{4})4(0,4,3)(\bar{0},\bar{4},\bar{3})2\bar{1}\bar{1}24 = (3,4,0)(\bar{3},\bar{4},\bar{0})32124.$ □

Since, $N^{(01214)} \geq N^{01214} =< e >$, and
$N(t_0t_1t_2t_1t_4)^{\{(1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0})\}} = 32124 \Rightarrow (1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0}) \in N^{(01214)}$
$\Rightarrow N^{(01214)} =< (1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0}) >= \{e, (1,2)(3,0)(\bar{1},\bar{2})(\bar{3},\bar{0})\}$.

Therefore, the number of single cosets in the double coset [01214] are at most

$\frac{|N|}{|N^{(01214)}|} = \frac{5!}{2} = \frac{120}{2} = 60$. Each single coset has two different names.

By conjugating ($01214 \sim 32124$) with the elements of transversal of $N^{(01214)}$, we obtain the sixty single cosets of double coset [01214]:
$01214 \sim 32124, 12320 \sim 43230, 23431 \sim 04341, 34042 \sim 10402, 12423 \sim 04243,$
$40103 \sim 21013, 24041 \sim 30401, 24340 \sim 13430, 23034 \sim 10304, 40302 \sim 13032,$
$01312 \sim 43132, 30102 \sim 41012, 43031 \sim 20301, 34140 \sim 21410, 03134 \sim 21314,$
$20104 \sim 31014, 01413 \sim 24143, 12024 \sim 30204, 41213 \sim 02123, 04142 \sim 31412,$
$40201 \sim 32021, 31210 \sim 42120, 10203 \sim 42023, 14240 \sim 32420, 23130 \sim 41310,$

$13234 \sim 02324, 02421 \sim 34241, 42321 \sim 03231, 14342 \sim 03432, 14043 \sim 20403,$
$\bar{0}\bar{1}\bar{2}\bar{1}\bar{3} \sim \bar{4}\bar{2}\bar{1}\bar{2}\bar{3}, \bar{1}\bar{2}\bar{4}\bar{2}\bar{0} \sim \bar{3}\bar{4}\bar{2}\bar{4}\bar{0}, \bar{1}\bar{2}\bar{3}\bar{2}\bar{4} \sim \bar{0}\bar{3}\bar{2}\bar{3}\bar{4}, \bar{2}\bar{4}\bar{3}\bar{4}\bar{1} \sim \bar{0}\bar{3}\bar{4}\bar{3}\bar{1}, \bar{2}\bar{3}\bar{0}\bar{3}\bar{1} \sim \bar{4}\bar{0}\bar{3}\bar{0}\bar{1},$
$\bar{2}\bar{3}\bar{4}\bar{3}\bar{0} \sim \bar{1}\bar{4}\bar{3}\bar{4}\bar{0}, \bar{4}\bar{3}\bar{0}\bar{3}\bar{2} \sim \bar{1}\bar{0}\bar{3}\bar{0}\bar{2}, \bar{3}\bar{0}\bar{4}\bar{3}\bar{2} \sim \bar{1}\bar{4}\bar{0}\bar{4}\bar{2}, \bar{3}\bar{4}\bar{1}\bar{4}\bar{2} \sim \bar{0}\bar{1}\bar{4}\bar{1}\bar{2}, \bar{3}\bar{4}\bar{0}\bar{4}\bar{1} \sim \bar{2}\bar{0}\bar{4}\bar{0}\bar{1},$
$\bar{3}\bar{0}\bar{1}\bar{0}\bar{4} \sim \bar{2}\bar{1}\bar{0}\bar{1}\bar{4}, \bar{0}\bar{4}\bar{1}\bar{4}\bar{3} \sim \bar{2}\bar{1}\bar{4}\bar{1}\bar{3}, \bar{4}\bar{1}\bar{0}\bar{1}\bar{3} \sim \bar{2}\bar{0}\bar{1}\bar{0}\bar{3}, \bar{4}\bar{0}\bar{2}\bar{0}\bar{3} \sim \bar{1}\bar{2}\bar{0}\bar{2}\bar{3}, \bar{4}\bar{0}\bar{1}\bar{0}\bar{2} \sim \bar{3}\bar{1}\bar{0}\bar{1}\bar{2},$
$\bar{2}\bar{4}\bar{0}\bar{4}\bar{3} \sim \bar{1}\bar{0}\bar{4}\bar{0}\bar{3}, \bar{4}\bar{1}\bar{2}\bar{1}\bar{0} \sim \bar{3}\bar{2}\bar{1}\bar{2}\bar{0}, \bar{1}\bar{0}\bar{2}\bar{0}\bar{4} \sim \bar{3}\bar{2}\bar{0}\bar{2}\bar{4}, \bar{0}\bar{2}\bar{1}\bar{2}\bar{4} \sim \bar{3}\bar{1}\bar{2}\bar{1}\bar{4}, \bar{0}\bar{1}\bar{3}\bar{1}\bar{4} \sim \bar{2}\bar{3}\bar{1}\bar{3}\bar{4},$
$\bar{2}\bar{1}\bar{3}\bar{1}\bar{0} \sim \bar{4}\bar{3}\bar{1}\bar{3}\bar{0}, \bar{0}\bar{2}\bar{3}\bar{2}\bar{1} \sim \bar{4}\bar{3}\bar{2}\bar{3}\bar{1}, \bar{1}\bar{3}\bar{2}\bar{3}\bar{0} \sim \bar{4}\bar{2}\bar{3}\bar{2}\bar{0}, \bar{0}\bar{3}\bar{1}\bar{3}\bar{2} \sim \bar{4}\bar{1}\bar{3}\bar{1}\bar{2}, \bar{3}\bar{0}\bar{2}\bar{0}\bar{1} \sim \bar{4}\bar{2}\bar{0}\bar{2}\bar{1},$
$\bar{3}\bar{2}\bar{4}\bar{2}\bar{1} \sim \bar{0}\bar{4}\bar{2}\bar{4}\bar{1}, \bar{1}\bar{3}\bar{4}\bar{3}\bar{2} \sim \bar{0}\bar{4}\bar{3}\bar{4}\bar{2}, \bar{2}\bar{4}\bar{1}\bar{4}\bar{0} \sim \bar{3}\bar{1}\bar{4}\bar{1}\bar{0}, \bar{1}\bar{4}\bar{2}\bar{4}\bar{3} \sim \bar{0}\bar{2}\bar{4}\bar{2}\bar{3}, \bar{2}\bar{0}\bar{3}\bar{0}\bar{4} \sim \bar{1}\bar{3}\bar{0}\bar{3}\bar{4}.$

The orbits of $N^{(01214)}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{4\}$, $\{\bar{4}\}$, $\{1,2\}$, $\{\bar{1},\bar{2}\}$,$\{3,0\}$ and $\{\bar{3},\bar{0}\}$. Now, take one representative from each orbit, namely 4, $\bar{4}$, 1, $\bar{1}$, 0 and $\bar{0}$, then find the double cosets to which $Nt_0t_1t_2t_1t_4t_4$, $Nt_0t_1t_2t_1t_4\bar{t_4}$, $Nt_0t_1t_2t_1t_4t_1$, $Nt_0t_1t_2t_1t_4\bar{t_1}$, $Nt_0t_1t_2t_1t_4t_0$ and $Nt_0t_1t_2t_1t_4\bar{t_0}$ belong. However,

$Nt_0t_1t_2t_1t_4t_4 = Nt_0t_1t_2t_1\bar{t_4} \in [0121\bar{4}]$.
$Nt_0t_1t_2t_1t_4\bar{t_4} = Nt_0t_1t_2t_1 \in [0121]$.
$Nt_0t_1t_2t_1t_4t_1 = Nt_0t_1t_2t_1t_4t_1 \in [01023]$ **(By Lemma-70)**.
$Nt_0t_1t_2t_1t_4\bar{t_1} = Nt_0t_1t_2t_1t_4\bar{t_1} \in [0\bar{1}\bar{2}\bar{3}]$ **(By Lemma-71)**.
$Nt_0t_1t_2t_1t_4t_0 = Nt_0t_1t_2t_1t_4t_0 \in [01203]$ **(By Lemma-72)**.
$Nt_0t_1t_2t_1t_4\bar{t_0} = Nt_0t_1t_2t_1t_4\bar{t_0} \in [0121]$ **(By Lemma-73)**.

Consider the double coset [01203]

$N^{(01203)} = N^{01203} =< e >$
Therefore, the number of single cosets in the double coset [01203] are at most

$\frac{|N|}{|N^{(01203)}|}=\frac{5!}{1} = \frac{120}{1} = 120$. Each single coset has only one name.

$\Rightarrow T \cong S_5$.
Now, conjugating (01203) with elements of transversal of $N^{(01203)}$, we obtain the following 120 single cosets of double coset [01203]:
01203, 31234, 41240, 41042, 31032, 21024, 21423, 31430, 01402, 01304, 41342, 21320,
12314, 42340, 02301, 02104, 42143, 32130, 32034, 42041, 12013, 12410, 02403, 32431,
40142, 20123, 30134, 30432, 20421, 10413, 10312, 20324, 40341, 40243, 30231, 10214,
23420, 03401, 13412, 13210, 03204, 43241, 43140, 03102, 23124, 23021, 13014, 43042,

$34031, 14012, 24023, 24321, 14310, 04302, 04201, 14213, 34230, 34132, 24120, 04103,$
$\bar{0}\bar{1}\bar{2}\bar{0}\bar{4}, \bar{4}\bar{1}\bar{2}\bar{4}\bar{3}, \bar{3}\bar{1}\bar{2}\bar{3}\bar{0}, \bar{3}\bar{1}\bar{0}\bar{3}\bar{4}, \bar{4}\bar{1}\bar{0}\bar{4}\bar{2}, \bar{2}\bar{1}\bar{0}\bar{2}\bar{3}, \bar{2}\bar{1}\bar{3}\bar{2}\bar{4}, \bar{4}\bar{1}\bar{3}\bar{4}\bar{0}, \bar{0}\bar{1}\bar{3}\bar{0}\bar{2}, \bar{0}\bar{1}\bar{4}\bar{0}\bar{3}, \bar{3}\bar{1}\bar{4}\bar{3}\bar{2}, \bar{2}\bar{1}\bar{4}\bar{2}\bar{0},$
$\bar{1}\bar{2}\bar{4}\bar{1}\bar{3}, \bar{3}\bar{2}\bar{4}\bar{3}\bar{0}, \bar{0}\bar{2}\bar{4}\bar{0}\bar{1}, \bar{0}\bar{2}\bar{1}\bar{0}\bar{3}, \bar{3}\bar{2}\bar{1}\bar{3}\bar{4}, \bar{4}\bar{2}\bar{1}\bar{4}\bar{0}, \bar{4}\bar{2}\bar{0}\bar{4}\bar{3}, \bar{3}\bar{2}\bar{0}\bar{3}\bar{1}, \bar{1}\bar{2}\bar{0}\bar{1}\bar{4}, \bar{1}\bar{2}\bar{3}\bar{1}\bar{0}, \bar{0}\bar{2}\bar{3}\bar{0}\bar{4}, \bar{4}\bar{2}\bar{3}\bar{4}\bar{1},$
$\bar{4}\bar{0}\bar{1}\bar{4}\bar{3}, \bar{3}\bar{0}\bar{1}\bar{3}\bar{2}, \bar{2}\bar{0}\bar{1}\bar{2}\bar{4}, \bar{2}\bar{0}\bar{4}\bar{2}\bar{3}, \bar{3}\bar{0}\bar{4}\bar{3}\bar{1}, \bar{1}\bar{0}\bar{4}\bar{1}\bar{2}, \bar{1}\bar{0}\bar{2}\bar{1}\bar{3}, \bar{3}\bar{0}\bar{2}\bar{3}\bar{4}, \bar{4}\bar{0}\bar{2}\bar{4}\bar{1}, \bar{4}\bar{0}\bar{3}\bar{4}\bar{2}, \bar{2}\bar{0}\bar{3}\bar{2}\bar{1}, \bar{1}\bar{0}\bar{3}\bar{1}\bar{4},$
$\bar{2}\bar{4}\bar{3}\bar{2}\bar{0}, \bar{0}\bar{4}\bar{3}\bar{0}\bar{1}, \bar{1}\bar{4}\bar{3}\bar{1}\bar{2}, \bar{1}\bar{4}\bar{2}\bar{1}\bar{0}, \bar{0}\bar{4}\bar{2}\bar{0}\bar{3}, \bar{3}\bar{4}\bar{2}\bar{3}\bar{1}, \bar{3}\bar{4}\bar{1}\bar{3}\bar{0}, \bar{0}\bar{4}\bar{1}\bar{0}\bar{2}, \bar{2}\bar{4}\bar{1}\bar{2}\bar{3}, \bar{2}\bar{4}\bar{0}\bar{2}\bar{1}, \bar{1}\bar{4}\bar{0}\bar{1}\bar{3}, \bar{3}\bar{4}\bar{0}\bar{3}\bar{2},$
$\bar{2}\bar{3}\bar{0}\bar{2}\bar{4}, \bar{4}\bar{3}\bar{0}\bar{4}\bar{1}, \bar{1}\bar{3}\bar{0}\bar{1}\bar{2}, \bar{1}\bar{3}\bar{2}\bar{1}\bar{4}, \bar{4}\bar{3}\bar{2}\bar{4}\bar{0}, \bar{0}\bar{3}\bar{2}\bar{0}\bar{1}, \bar{0}\bar{3}\bar{1}\bar{0}\bar{4}, \bar{4}\bar{3}\bar{1}\bar{4}\bar{2}, \bar{2}\bar{3}\bar{1}\bar{2}\bar{0}, \bar{2}\bar{3}\bar{4}\bar{2}\bar{1}, \bar{1}\bar{3}\bar{4}\bar{1}\bar{0}, \bar{0}\bar{3}\bar{4}\bar{0}\bar{2}.$

The orbits of $N^{(01203)}$ on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, $\{\bar{3}\}$ and $\{\bar{4}\}$. Now, take one representative from each orbit and find the double cosets to which $Nt_0t_1t_2t_0t_3t_0$, $Nt_0t_1t_2t_0t_3t_1$, $Nt_0t_1t_2t_0t_3t_2$, $Nt_0t_1t_2t_0t_3t_3$, $Nt_0t_1t_2t_0t_3t_4$, $Nt_0t_1t_2t_0t_3t_{\bar{0}}$, $Nt_0t_1t_2t_0t_3t_{\bar{1}}$, $Nt_0t_1t_2t_0t_3t_{\bar{2}}$, $Nt_0t_1t_2t_0t_3t_{\bar{3}}$ and $Nt_0t_1t_2t_0t_3t_{\bar{4}}$ belong. However,

$Nt_0t_1t_2t_0t_3t_0 = Nt_0t_1t_2t_0t_3t_0 \in [012\bar{3}\bar{0}]$ **(By Lemma-74).**
$Nt_0t_1t_2t_0t_3t_1 = Nt_0t_1t_2t_0t_3t_1 \in [0120]$ **(By Lemma-75).**
$Nt_0t_1t_2t_0t_3t_2 = Nt_0t_1t_2t_0t_3t_2 \in [0123]$ **(By Lemma-76).**
$Nt_0t_1t_2t_0t_3t_3 = Nt_0t_1t_2t_0t_{\bar{3}} \in [0120\bar{3}]$.
$Nt_0t_1t_2t_0t_3t_4 = Nt_0t_1t_2t_0t_3t_4 \in [0121]$ **(By Lemma-77).**
$Nt_0t_1t_2t_0t_3t_{\bar{0}} = Nt_0t_1t_2t_0t_3t_{\bar{0}} \in [01023]$ **(By Lemma-78).**
$Nt_0t_1t_2t_0t_3t_{\bar{1}} = Nt_0t_1t_2t_0t_3t_{\bar{1}} \in [01201]$ **(By Lemma-79).**
$Nt_0t_1t_2t_0t_3t_{\bar{2}} = Nt_0t_1t_2t_0t_3t_{\bar{2}} \in [01204]$ **(By Lemma-80).**
$Nt_0t_1t_2t_0t_3t_{\bar{3}} = Nt_0t_1t_2t_0 \in [0120]$.
$Nt_0t_1t_2t_0t_3t_{\bar{4}} = Nt_0t_1t_2t_0t_3t_{\bar{4}} \in [01214]$ **(By Lemma-81).**

Consider the double coset $012\bar{3}\bar{0}$
Claim: $012\bar{3}\bar{0} \sim 034\bar{1}\bar{0}$

Proof. By our relation-I, $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ conjugating with element of N, we obtain the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,2,3,4,0)(\bar{1},\bar{2},\bar{3},\bar{4},\bar{0})\}} = (\bar{0}1)^{\{(1,2,3,4,0)(\bar{1},\bar{2},\bar{3},\bar{4},\bar{0})\}}$
  $\Rightarrow (3,4,0)(\bar{3},\bar{4},\bar{0})\bar{2}1 = \bar{1}2$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{3},4,\bar{0})(2,\bar{2})(3,\bar{4},0,\bar{1})\}} = (\bar{0}1)^{\{(1,\bar{3},4,\bar{0})(2,\bar{2})(3,\bar{4},0,\bar{1})\}}$

$\Rightarrow (2,4,0)(\bar{2},\bar{4},\bar{0})3\bar{1} = 1\bar{3}$

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0})\}} = (\bar{0}1)^{\{(1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0})\}}$
  $\Rightarrow (2,1,0)(\bar{2},\bar{1},\bar{0})\bar{3}4 = \bar{4}3$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{0},3,\bar{2})(2,\bar{1},0,\bar{3})(4,\bar{4})\}} = (\bar{0}1)^{\{(1,\bar{0},3,\bar{2})(2,\bar{1},0,\bar{3})(4,\bar{4})\}}$
  $\Rightarrow (1,2,4)(\bar{1},\bar{2},\bar{4})0\bar{3} = 3\bar{0}$

From the relations above, we have
$012\bar{3}\bar{0} = 0\bar{1}\underline{12}\bar{3}\bar{0} = 0\bar{1}(3,4,0)(\bar{3},\bar{4},\bar{0})\bar{2}1\bar{3}\bar{0} = (3,4,0)(\bar{3},\bar{4},\bar{0})3\bar{1}\bar{2}\underline{1\bar{3}}\bar{0}$
$= (3,4,0)(\bar{3},\bar{4},\bar{0})3\bar{1}\bar{2}(2,4,0)(\bar{2},\bar{4},\bar{0})3\bar{1}\bar{0} = (3,0)(4,2)(\bar{3},\bar{0})(\bar{4},\bar{2})3\bar{1}\underline{\bar{4}3}\bar{1}\bar{0}$
$= (3,0)(4,2)(\bar{3},\bar{0})(\bar{4},\bar{2})3\bar{1}(2,1,0)(\bar{2},\bar{1},\bar{0})\bar{3}4\bar{1}\bar{0} = (3,2,4,1,0)(\bar{3},\bar{2},\bar{4},\bar{1},\bar{0})\underline{3\bar{0}}\bar{3}4\bar{1}\bar{0}$
$= (3,2,4,1,0)(\bar{3},\bar{2},\bar{4},\bar{1},\bar{0})(1,2,4)(\bar{1},\bar{2},\bar{4})0\bar{3}\bar{3}4\bar{1}\bar{0} = (3,4,2,1,0)(\bar{3},\bar{4},\bar{2},\bar{1},\bar{0})034\bar{1}\bar{0}.$ □

Since, $N^{(012\bar{3}\bar{0})} \geq N^{012\bar{3}\bar{0}} =< e >$, and
$N(t_0t_1t_2\bar{t_1}\bar{t_4})^{\{(1,3)(2,4)(\bar{1},\bar{3})(\bar{2},\bar{4})\}} = 034\bar{1}\bar{0} \Rightarrow (1,3)(2,4)(\bar{1},\bar{3})(\bar{2},\bar{4}) \in N^{(012\bar{3}\bar{0})}$
$\Rightarrow N^{(012\bar{3}\bar{0})} =< (1,3)(2,4)(\bar{1},\bar{3})(\bar{2},\bar{4}) >= \{e, (1,3)(2,4)(\bar{1},\bar{3})(\bar{2},\bar{4})\}$.

Therefore, the number of single cosets in the double coset $[012\bar{3}\bar{0}]$ are at most

$\frac{|N|}{|N^{(012\bar{3}\bar{0})}|} = \frac{5!}{2} = \frac{120}{2} = 60$. Each single coset has two different names.

By conjugating $(012\bar{3}\bar{0} \sim 034\bar{1}\bar{0})$ with the elements of transversal of $N^{(012\bar{3}\bar{0})}$, we obtain the sixty single cosets of double coset $[012\bar{3}\bar{0}]$:
$012\bar{3}\bar{0} \sim 034\bar{1}\bar{0}, 123\bar{4}\bar{1} \sim 140\bar{2}\bar{1}, 234\bar{0}\bar{2} \sim 201\bar{3}\bar{2}, 340\bar{1}\bar{3} \sim 312\bar{4}\bar{3}, 124\bar{0}\bar{1} \sim 103\bar{2}\bar{1},$
$401\bar{2}\bar{4} \sim 423\bar{0}\bar{4}, 240\bar{3}\bar{2} \sim 231\bar{4}\bar{2}, 243\bar{1}\bar{2} \sim 210\bar{4}\bar{2}, 230\bar{1}\bar{2} \sim 214\bar{3}\bar{2}, 412\bar{0}\bar{4} \sim 403\bar{1}\bar{4},$
$431\bar{0}\bar{4} \sim 402\bar{3}\bar{4}, 301\bar{4}\bar{3} \sim 342\bar{0}\bar{3}, 430\bar{2}\bar{4} \sim 421\bar{3}\bar{4}, 304\bar{2}\bar{3} \sim 321\bar{0}\bar{3}, 341\bar{2}\bar{3} \sim 320\bar{4}\bar{3},$
$024\bar{3}\bar{0} \sim 031\bar{2}\bar{0}, 324\bar{1}\bar{3} \sim 310\bar{2}\bar{3}, 014\bar{2}\bar{0} \sim 023\bar{4}\bar{0}, 302\bar{1}\bar{3} \sim 314\bar{0}\bar{3}, 013\bar{4}\bar{0} \sim 042\bar{1}\bar{0},$
$041\bar{3}\bar{0} \sim 032\bar{4}\bar{0}, 410\bar{3}\bar{4} \sim 432\bar{1}\bar{4}, 143\bar{0}\bar{1} \sim 102\bar{4}\bar{1}, 142\bar{3}\bar{1} \sim 130\bar{4}\bar{1}, 021\bar{4}\bar{0} \sim 043\bar{2}\bar{0},$
$134\bar{2}\bar{1} \sim 120\bar{3}\bar{1}, 413\bar{2}\bar{4} \sim 420\bar{1}\bar{4}, 213\bar{0}\bar{2} \sim 204\bar{1}\bar{2}, 203\bar{4}\bar{2} \sim 241\bar{0}\bar{2}, 132\bar{0}\bar{1} \sim 104\bar{3}\bar{1},$
$\bar{1}\bar{2}\bar{4}31 \sim \bar{1}\bar{3}\bar{0}21, \bar{0}\bar{1}\bar{2}40 \sim \bar{0}\bar{4}\bar{3}10, \bar{1}\bar{2}\bar{3}01 \sim \bar{1}\bar{0}\bar{4}21, \bar{2}\bar{4}\bar{3}02 \sim \bar{2}\bar{0}\bar{1}42, \bar{2}\bar{3}\bar{0}42 \sim \bar{2}\bar{4}\bar{1}32,$
$\bar{2}\bar{3}\bar{4}12 \sim \bar{2}\bar{1}\bar{0}32, \bar{4}\bar{1}\bar{2}34 \sim \bar{4}\bar{3}\bar{0}14, \bar{3}\bar{0}\bar{4}13 \sim \bar{3}\bar{1}\bar{2}03, \bar{3}\bar{4}\bar{1}03 \sim \bar{3}\bar{0}\bar{2}43, \bar{3}\bar{4}\bar{0}23 \sim \bar{3}\bar{2}\bar{1}43,$
$\bar{3}\bar{0}\bar{1}23 \sim \bar{3}\bar{2}\bar{4}03, \bar{0}\bar{2}\bar{3}40 \sim \bar{0}\bar{4}\bar{1}20, \bar{4}\bar{1}\bar{0}24 \sim \bar{4}\bar{2}\bar{3}14, \bar{4}\bar{0}\bar{2}14 \sim \bar{4}\bar{1}\bar{3}04, \bar{2}\bar{4}\bar{0}12 \sim \bar{2}\bar{1}\bar{3}42,$
$\bar{1}\bar{3}\bar{4}01 \sim \bar{1}\bar{0}\bar{2}31, \bar{0}\bar{2}\bar{1}30 \sim \bar{0}\bar{3}\bar{4}20, \bar{0}\bar{1}\bar{3}20 \sim \bar{0}\bar{2}\bar{4}10, \bar{4}\bar{0}\bar{3}24 \sim \bar{4}\bar{2}\bar{1}04, \bar{4}\bar{3}\bar{1}24 \sim \bar{4}\bar{2}\bar{0}34,$
$\bar{1}\bar{3}\bar{2}41 \sim \bar{1}\bar{4}\bar{0}31, \bar{0}\bar{1}\bar{4}30 \sim \bar{0}\bar{3}\bar{2}10, \bar{0}\bar{3}\bar{1}40 \sim \bar{0}\bar{4}\bar{2}30, \bar{3}\bar{4}\bar{2}13 \sim \bar{3}\bar{1}\bar{0}43, \bar{1}\bar{4}\bar{3}21 \sim \bar{1}\bar{2}\bar{0}41,$

$\bar3\bar1\bar423 \sim \bar3\bar2\bar013, \bar1\bar4\bar201 \sim \bar1\bar0\bar341, \bar2\bar0\bar312 \sim \bar2\bar1\bar402, \bar2\bar0\bar432 \sim \bar2\bar3\bar102, \bar4\bar0\bar134 \sim \bar4\bar3\bar204.$

The orbits of $N^{(0123\bar3\bar0)}$ on $\{0,1,2,3,4,\bar0,\bar1,\bar2,\bar3,\bar4\}$ are $\{0\}$, $\{\bar0\}$, $\{1,3\}$, $\{\bar1,\bar3\}$,$\{2,4\}$ and $\{\bar2,\bar4\}$. Now, take one representative from each orbit, namely 0, $\bar0$, 1, $\bar1$, 2 and $\bar2$, then find the double cosets to which $Nt_0t_1t_2\bar{t_3}\bar{t_0}t_0$, $Nt_0t_1t_2\bar{t_3}\bar{t_0}\bar{t_0}$, $Nt_0t_1t_2\bar{t_3}\bar{t_0}t_1$, $Nt_0t_1t_2\bar{t_3}\bar{t_0}\bar{t_1}$, $Nt_0t_1t_2\bar{t_3}\bar{t_0}t_2$ and $Nt_0t_1t_2\bar{t_3}\bar{t_0}\bar{t_2}$ belong. However,

$Nt_0t_1t_2\bar{t_3}\bar{t_0}t_0 = Nt_0t_1t_2\bar{t_3} \in [012\bar3]$.
$Nt_0t_1t_2\bar{t_3}\bar{t_0}\bar{t_0} = Nt_0t_1t_2\bar{t_3}t_0 \in [0\bar1\bar23]$.
$Nt_0t_1t_2\bar{t_3}\bar{t_0}t_1 = Nt_0t_1t_2\bar{t_3}\bar{t_0}t_1 \in [01\bar2\bar0]$ **(By Lemma-82).**
$Nt_0t_1t_2\bar{t_3}\bar{t_0}\bar{t_1} = Nt_0t_1t_2\bar{t_3}\bar{t_0}\bar{t_1} \in [012\bar3\bar0]$ **(By Lemma-83).**
$Nt_0t_1t_2\bar{t_3}\bar{t_0}t_2 = Nt_0t_1t_2\bar{t_3}\bar{t_0}t_2 \in [01203]$ **(By Lemma-84).**
$Nt_0t_1t_2\bar{t_3}\bar{t_0}\bar{t_2} = Nt_0t_1t_2\bar{t_3}\bar{t_0}\bar{t_2} \in [01023]$ **(By Lemma-85).**

Consider the double coset $[0\bar1\bar2\bar34]$
Claim: $0\bar1\bar2\bar34 \sim 2\bar1\bar0\bar43 \sim \bar2310\bar4$

Proof. By our relation-I, $(2,3,4)(\bar2,\bar3,\bar4)\bar10 = \bar01$ conjugating with element of N, we obtain the following relations:

- $((2,3,4)(\bar2,\bar3,\bar4)\bar10)^{\{(1,\bar1)(2,\bar0)(3,\bar3)(4,\bar4)(0,\bar2)\}} = (\bar01)^{\{(1,\bar1)(2,\bar0)(3,\bar3)(4,\bar4)(0,\bar2)\}}$
  $\Rightarrow (0,3,4)(\bar0,\bar3,\bar4)1\bar2 = 2\bar1$
- $((2,3,4)(\bar2,\bar3,\bar4)\bar10)^{\{(1,3)(4,0)(\bar1,\bar3)(\bar4,\bar0)\}} = (\bar01)^{\{(1,3)(4,0)(\bar1,\bar3)(\bar4,\bar0)\}}$
  $\Rightarrow (2,1,0)(\bar2,\bar1,\bar0)\bar34 = \bar43$
- $((2,3,4)(\bar2,\bar3,\bar4)\bar10)^{\{(1,3)(0,2)(\bar1,\bar3)(\bar0,\bar2)\}} = (\bar01)^{\{(1,3)(0,2)(\bar1,\bar3)(\bar0,\bar2)\}}$
  $\Rightarrow (0,1,4)(\bar0,\bar1,\bar4)\bar32 = \bar23$

From the relations above, we have
1. $0\bar1\bar2\underline{\bar34} = 0\bar1\bar2(2,0,1)(\bar2,\bar0,\bar1)\bar43 = (2,0,1)(\bar2,\bar0,\bar1)\underline{\bar1\bar2}\bar0\bar43$
$= (2,0,1)(\bar2,\bar0,\bar1)(0,3,4)(\bar0,\bar4,\bar3)2\bar1\bar0\bar43 = (2,4,3,0,1)(\bar2,\bar4,\bar3,\bar0,\bar1)2\bar1\bar0\bar43.$

2. By our relation-II, $(0,3,1,4,2)(\bar0,\bar3,\bar1,\bar4,\bar2)2104 = \bar0\bar1\bar2\bar3$
$$\Leftrightarrow \bar0(0,3,1,4,2)(\bar0,\bar3,\bar1,\bar4,\bar2)2104 = \bar0\bar0\bar1\bar2\bar3$$

$$\Leftrightarrow (0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})\bar{3}2104 = 0\bar{1}\bar{2}\bar{3}$$
$$\Leftrightarrow (0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})\bar{3}21044 = 0\bar{1}\bar{2}\bar{3}$$
$$\Leftrightarrow (0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})\underline{\bar{3}2}10\bar{4} = 0\bar{1}\bar{2}\bar{3}4$$
$$\Leftrightarrow (0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})(0,4,1)(\bar{0},\bar{4},\bar{1})\bar{2}310\bar{4} = 0\bar{1}\bar{2}\bar{3}4$$
$$\Leftrightarrow (0,3)(4,2)(\bar{0},\bar{3})(\bar{4},\bar{2})\bar{2}310\bar{4} = 0\bar{1}\bar{2}\bar{3}4 \qquad \square$$

Since, $N^{(0\bar{1}\bar{2}\bar{3}4)} \geq N^{0\bar{1}\bar{2}\bar{3}4} =< e >$, and
$N(t_0\bar{t_1}\bar{t_2}\bar{t_3}t_4)^{\{(2,0)(3,4)(\bar{2},\bar{0})(\bar{3},\bar{4})\}} = 2\bar{1}\bar{0}\bar{4}3 \Rightarrow (2,0)(3,4)(\bar{2},\bar{0})(\bar{3},\bar{4}) \in N^{(0\bar{1}\bar{2}\bar{3}4)}$
$N(t_0\bar{t_1}\bar{t_2}\bar{t_3}t_4)^{\{(1,\bar{3},0,\bar{2})(2,\bar{1},3,\bar{0})(3,\bar{4})\}} = \bar{2}310\bar{4} \Rightarrow (1,\bar{3},0,\bar{2})(2,\bar{1},3,\bar{0})(3,\bar{4}) \in N^{(0\bar{1}\bar{2}\bar{3}4)}$
$\Rightarrow N^{(0\bar{1}\bar{2}\bar{3}4)} =< (2,0)(3,4)(\bar{2},\bar{0})(\bar{3},\bar{4}), (1,\bar{3},0,\bar{2})(2,\bar{1},3,\bar{0})(3,\bar{4}) >$
$= \{e, (1,\bar{3},0,\bar{2})(2,\bar{1},3,\bar{0})(4,\bar{4}), (1,\bar{2},0,\bar{3})(2,\bar{0},3,\bar{1})(4,\bar{4}), (1,\bar{4},3,\bar{2})(2,\bar{1},4,\bar{3})(0,\bar{0}),$
$(1,0)(2,3)(\bar{1},\bar{0})(\bar{2},\bar{3}), (1,\bar{0},4,\bar{3})(2,\bar{2})(3,\bar{1},0,\bar{4}), (1,4,0,2,3)(\bar{1},\bar{4},\bar{0},\bar{2},\bar{3}),$
$(1,2,4,3,0)(\bar{1},\bar{2},\bar{4},\bar{3},\bar{0}), (1,3,2,0,4)(\bar{1},\bar{3},\bar{2},\bar{0},\bar{4}), (1,\bar{1})(2,\bar{4},0,\bar{3})(3,\bar{2},4,\bar{0}),$
$(2,0)(3,4)(\bar{2},\bar{0})(\bar{3},\bar{4}), (1,\bar{3},4,\bar{0})(2,\bar{2})(3,\bar{4},0,\bar{1}), (1,\bar{2},3,\bar{4})(2,\bar{3},4,\bar{1})(0,\bar{0}),$
$(1,\bar{4},2,\bar{0})(3,\bar{3})(4,\bar{2},0,\bar{1}), (1,0,3,4,2)(\bar{1},\bar{0},\bar{3},\bar{4},\bar{2}), (1,\bar{0},2,\bar{4})(3,\bar{3})(4,\bar{1},0,\bar{2}),$
$(1,4)(3,0)(\bar{1},\bar{4})(\bar{3},\bar{0}), (1,2)(4,0)(\bar{1},\bar{2})(\bar{4},\bar{0}), (1,3)(2,4)(\bar{1},\bar{3})(\bar{2},\bar{4}),$
$(1,\bar{1})(2,\bar{3},0,\bar{4})(3,\bar{0},4,\bar{2})\}$.

Now, take $Nt_0\bar{t_1}\bar{t_2}\bar{t_3}t_4$ conjugates with elements of $N^{(0\bar{1}\bar{2}\bar{3}4)}$, we obtain the following relation:

$$0\bar{1}\bar{2}\bar{3}4 \sim 2\bar{1}\bar{0}\bar{4}3 \sim 0\bar{3}\bar{4}\bar{1}2 \sim 1\bar{0}\bar{3}\bar{2}4 \sim 2\bar{4}\bar{3}\bar{1}0 \sim 3\bar{4}\bar{2}\bar{0}1 \sim 1\bar{2}\bar{4}\bar{0}3 \sim 4\bar{3}\bar{0}\bar{2}1 \sim 3\bar{0}\bar{1}\bar{4}2 \sim 4\bar{2}\bar{1}\bar{3}0 \sim$$
$$\bar{4}130\bar{2} \sim \bar{3}201\bar{4} \sim \bar{4}021\bar{3} \sim \bar{2}043\bar{1} \sim \bar{0}234\bar{1} \sim \bar{3}142\bar{0} \sim \bar{2}310\bar{4} \sim \bar{1}403\bar{2} \sim \bar{1}324\bar{0} \sim \bar{0}412\bar{3}.$$

Therefore, the number of single cosets in the double coset $[0\bar{1}\bar{2}\bar{3}4]$ are at most

$\frac{|N|}{|N^{(01234)}|} = \frac{5!}{20} = \frac{120}{20} = 6$. Each single coset has twenty different names.

By conjugating

$$(0\bar{1}\bar{2}\bar{3}4 \sim 2\bar{1}\bar{0}\bar{4}3 \sim 0\bar{3}\bar{4}\bar{1}2 \sim 1\bar{0}\bar{3}\bar{2}4 \sim 2\bar{4}\bar{3}\bar{1}0 \sim 3\bar{4}\bar{2}\bar{0}1 \sim 1\bar{2}\bar{4}\bar{0}3 \sim 4\bar{3}\bar{0}\bar{2}1 \sim 3\bar{0}\bar{1}\bar{4}2 \sim 4\bar{2}\bar{1}\bar{3}0 \sim$$
$$\bar{4}130\bar{2} \sim \bar{3}201\bar{4} \sim \bar{4}021\bar{3} \sim \bar{2}043\bar{1} \sim \bar{0}234\bar{1} \sim \bar{3}142\bar{0} \sim \bar{2}310\bar{4} \sim \bar{1}403\bar{2} \sim \bar{1}324\bar{0} \sim \bar{0}412\bar{3})$$

with the elements in the transversal of $N^{(0\bar{1}\bar{2}\bar{3}4)}$, we obtain the six single cosets of double coset $[0\bar{1}\bar{2}\bar{3}4]$:

1. $0\bar{1}\bar{2}\bar{3}4 \sim 2\bar{1}\bar{0}\bar{4}3 \sim 0\bar{3}\bar{4}\bar{1}2 \sim 1\bar{0}\bar{3}\bar{2}4 \sim 2\bar{4}\bar{3}\bar{1}0 \sim 3\bar{4}\bar{2}\bar{0}1 \sim 1\bar{2}\bar{4}\bar{0}3 \sim 4\bar{3}\bar{0}\bar{2}1 \sim 3\bar{0}\bar{1}\bar{4}2 \sim$

$4\bar{2}\bar{1}\bar{3}0 \sim \bar{4}130\bar{2} \sim \bar{3}201\bar{4} \sim \bar{4}021\bar{3} \sim \bar{2}043\bar{1} \sim \bar{0}234\bar{1} \sim \bar{3}142\bar{0} \sim \bar{2}310\bar{4} \sim \bar{1}403\bar{2} \sim \bar{1}324\bar{0} \sim \bar{0}412\bar{3}$.

2. $1\bar{2}\bar{3}\bar{4}0 \sim 3\bar{2}\bar{1}\bar{0}4 \sim 1\bar{4}\bar{0}\bar{2}3 \sim 2\bar{1}\bar{4}\bar{3}0 \sim 3\bar{0}\bar{4}\bar{2}1 \sim 4\bar{0}\bar{3}\bar{1}2 \sim 2\bar{3}\bar{0}\bar{1}4 \sim 0\bar{4}\bar{1}\bar{3}2 \sim 4\bar{1}\bar{2}\bar{0}3 \sim 0\bar{3}\bar{2}\bar{4}1 \sim \bar{0}241\bar{3} \sim \bar{4}312\bar{0} \sim \bar{0}132\bar{4} \sim \bar{3}104\bar{2} \sim \bar{1}340\bar{2} \sim \bar{4}203\bar{1} \sim \bar{3}421\bar{0} \sim \bar{2}014\bar{3} \sim \bar{2}430\bar{1} \sim \bar{1}023\bar{4}$

3. $3\bar{1}\bar{4}\bar{0}2 \sim 4\bar{2}\bar{0}\bar{1}3 \sim 3\bar{0}\bar{2}\bar{1}4 \sim 2\bar{0}\bar{3}\bar{4}1 \sim 0\bar{2}\bar{4}\bar{3}1 \sim 4\bar{1}\bar{3}\bar{2}0 \sim 2\bar{4}\bar{1}\bar{0}3 \sim 1\bar{3}\bar{0}\bar{4}2 \sim 1\bar{4}\bar{2}\bar{3}0 \sim 0\bar{3}\bar{1}\bar{2}4 \sim \bar{0}124\bar{3} \sim \bar{2}103\bar{4} \sim \bar{0}431\bar{2} \sim \bar{1}042\bar{3} \sim \bar{2}341\bar{0} \sim \bar{4}320\bar{1} \sim \bar{1}230\bar{4} \sim \bar{3}402\bar{1} \sim \bar{4}013\bar{2} \sim \bar{3}214\bar{0}$

4. $2\bar{3}\bar{4}\bar{0}1 \sim 4\bar{3}\bar{2}\bar{1}0 \sim 2\bar{0}\bar{1}\bar{3}4 \sim 3\bar{2}\bar{0}\bar{4}1 \sim 4\bar{1}\bar{0}\bar{3}2 \sim 0\bar{1}\bar{4}\bar{2}3 \sim 3\bar{4}\bar{1}\bar{2}0 \sim 1\bar{0}\bar{2}\bar{4}3 \sim 0\bar{2}\bar{3}\bar{1}4 \sim 1\bar{4}\bar{3}\bar{0}2 \sim \bar{1}302\bar{4} \sim \bar{0}423\bar{1} \sim \bar{1}243\bar{0} \sim \bar{4}210\bar{3} \sim \bar{2}401\bar{3} \sim \bar{0}314\bar{2} \sim \bar{4}032\bar{1} \sim \bar{3}120\bar{4} \sim \bar{3}041\bar{2} \sim \bar{2}134\bar{0}$

5. $3\bar{4}\bar{0}\bar{1}2 \sim 0\bar{2}\bar{1}\bar{4}3 \sim 3\bar{1}\bar{2}\bar{4}0 \sim 2\bar{1}\bar{3}\bar{0}4 \sim 1\bar{2}\bar{0}\bar{3}4 \sim 0\bar{4}\bar{3}\bar{2}1 \sim 2\bar{0}\bar{4}\bar{1}3 \sim 4\bar{3}\bar{1}\bar{0}2 \sim 4\bar{0}\bar{2}\bar{3}1 \sim 1\bar{3}\bar{4}\bar{2}0 \sim \bar{1}420\bar{3} \sim \bar{2}413\bar{0} \sim \bar{1}034\bar{2} \sim \bar{4}102\bar{3} \sim \bar{2}304\bar{1} \sim \bar{0}321\bar{4} \sim \bar{4}231\bar{0} \sim \bar{3}012\bar{4} \sim \bar{0}143\bar{2} \sim \bar{3}240\bar{1}$

6. $4\bar{0}\bar{1}\bar{2}3 \sim 1\bar{0}\bar{4}\bar{3}2 \sim 4\bar{2}\bar{3}\bar{0}1 \sim 0\bar{4}\bar{2}\bar{1}3 \sim 1\bar{3}\bar{2}\bar{0}4 \sim 2\bar{3}\bar{1}\bar{4}0 \sim 0\bar{1}\bar{3}\bar{4}2 \sim 3\bar{2}\bar{4}\bar{1}0 \sim 2\bar{4}\bar{0}\bar{3}1 \sim 3\bar{1}\bar{0}\bar{2}4 \sim \bar{3}024\bar{1} \sim \bar{2}140\bar{3} \sim \bar{3}410\bar{2} \sim \bar{1}432\bar{0} \sim \bar{4}123\bar{0} \sim \bar{2}031\bar{4} \sim \bar{1}204\bar{3} \sim \bar{0}342\bar{1} \sim \bar{0}213\bar{4} \sim \bar{4}301\bar{2}$

The double coset $N^{(0\bar{1}\bar{2}\bar{3}4)}$ acts transitively on $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$. Thus, $N^{(0\bar{1}\bar{2}\bar{3}4}$ has only a single orbit $\{0,1,2,3,4,\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$. Now, take one representative from the orbit, say $\bar{4}$, and find to which the double coset $Nt_0t_{\bar{1}}t_{\bar{2}}t_{\bar{3}}t_4t_{\bar{4}}$ belongs. However,
$Nt_0t_{\bar{1}}t_{\bar{2}}t_{\bar{3}}t_4t_{\bar{4}} = Nt_0t_{\bar{1}}t_{\bar{2}}t_{\bar{3}} \in [0\bar{1}\bar{2}\bar{3}]$
Therefore, all ten symmetric generators go back to the double coset $[0\bar{1}\bar{2}\bar{3}]$, and none of them extends.

Consider the double coset [010232]
Claim: $010232 \sim \bar{0}\bar{1}\bar{0}\bar{3}\bar{2}\bar{3} \sim 313404$

Proof. By our relation-I $(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0 = \bar{0}1$ and relation-II $(0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})2104 = \bar{0}\bar{1}\bar{2}\bar{3}$ conjugating with element of N, we obtain the following relations:

- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,4)(2,0)(\bar{1},\bar{4})(\bar{2},\bar{0})\}} = (\bar{0}1)^{\{(1,4)(2,0)(\bar{1},\bar{4})(\bar{2},\bar{0})\}}$
  $\Rightarrow (0,3,1)(\bar{0},\bar{3},\bar{1})\bar{4}2 = \bar{2}4$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{4},0,\bar{2})(2,\bar{1},4,\bar{0})(3,\bar{3})\}} = (\bar{0}1)^{\{(1,\bar{4},0,\bar{2})(2,\bar{1},4,\bar{0})(3,\bar{3})\}}$
  $\Rightarrow (1,3,0)(\bar{1},\bar{3},\bar{0})4\bar{2} = 2\bar{4}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{3},4,\bar{0})(2,\bar{2})(3,\bar{4},0,\bar{1})\}} = (\bar{0}1)^{\{(1,\bar{3},4,\bar{0})(2,\bar{2})(3,\bar{4},0,\bar{1})\}}$
  $\Rightarrow (2,4,0)(\bar{2},\bar{4},\bar{0})3\bar{1} = 1\bar{3}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,3)(0,2)(\bar{1},\bar{3})(\bar{0},\bar{2})\}} = (\bar{0}1)^{\{(1,3)(0,2)(\bar{1},\bar{3})(\bar{0},\bar{2})\}}$
  $\Rightarrow (0,1,4)(\bar{0},\bar{1},\bar{4})\bar{3}2 = \bar{2}3$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{3},0,\bar{2})(2,\bar{1},3,\bar{0})(4,\bar{4})\}} = (\bar{0}1)^{\{(1,\bar{3},0,\bar{2})(2,\bar{1},3,\bar{0})(4,\bar{4})\}}$
  $\Rightarrow (1,0,4)(\bar{1},\bar{0},\bar{4})3\bar{2} = 2\bar{3}$
- $((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}} = (\bar{0}1)^{\{(1,\bar{1})(2,\bar{2})(3,\bar{4})(4,\bar{3})(0,\bar{0})\}}$
  $\Rightarrow (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0} = 0\bar{1}$
- $((0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})2104)^{\{(1,\bar{0})(2,\bar{2})(3,\bar{3})(4,\bar{4})(0,\bar{1})\}} = (\bar{0}\bar{1}\bar{2}\bar{3})^{\{(1,\bar{0})(2,\bar{2})(3,\bar{3})(4,\bar{4})(0,\bar{1})\}}$
  $\Rightarrow (1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2})\bar{2}\bar{0}\bar{1}\bar{4} = 1023$
- $((0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})2104)^{\{(2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0})\}} = (\bar{0}\bar{1}\bar{2}\bar{3})^{\{(2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0})\}}$
  $\Rightarrow (4,2,1,0,3)(\bar{4},\bar{2},\bar{1},\bar{0},\bar{3})3140 = \bar{4}\bar{1}\bar{3}\bar{2}$

From the relations above, we have

1. $010232 = 0102\bar{3}\underline{\bar{3}2} = 0102\bar{3}(0,4,1)(\bar{0},\bar{4},\bar{1})\bar{2}3 = (0,4,1)(\bar{0},\bar{4},\bar{1})4042\bar{3}\bar{2}3$
$= (0,4,1)(\bar{0},\bar{4},\bar{1})4042\bar{3}2\underline{2\bar{3}}\bar{3} = (0,4,1)(\bar{0},\bar{4},\bar{1})4042\bar{3}2(1,0,4)(\bar{1},\bar{0},\bar{4})3\bar{2}\bar{3}$
$= (0,1,4)(\bar{0},\bar{1},\bar{4})1412\bar{3}23\bar{2}\bar{3} = (0,1,4)(\bar{0},\bar{1},\bar{4})1412\bar{3}\underline{2\bar{3}}3\bar{2}\bar{3}$
$= (0,1,4)(\bar{0},\bar{1},\bar{4})1412\bar{3}(1,0,4)(\bar{1},\bar{0},\bar{4})3\bar{2}\bar{3}\bar{2}\bar{3} = 0102\bar{3}3\bar{2}\bar{3}\bar{2}\bar{3} = 010\bar{3}\bar{2}\bar{3}$
$= 0\underline{1\bar{0}}\bar{0}\bar{3}\bar{2}\bar{3} = 0(2,3,4)(\bar{2},\bar{3},\bar{4})0\bar{1}\bar{0}\bar{3}\bar{2}\bar{3} = (2,3,4)(\bar{2},\bar{3},\bar{4})00\bar{1}\bar{0}\bar{3}\bar{2}\bar{3}$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}\bar{0}\bar{3}\bar{2}\bar{3}.$

2. $0\underline{1023}2 = 0(1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2})\overline{2014}2 = (1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2})4\overline{201}\underline{\bar{4}2}$
$= (1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2})4\overline{201}(0,1,3)(\bar{0},\bar{1},\bar{3})\bar{2}4(1,0,4,2,3)(\bar{1},\bar{0},\bar{4},\bar{2},\bar{3})\underline{4\bar{2}}\overline{132}4$
$= (1,0,4,2,3)(\bar{1},\bar{0},\bar{4},\bar{2},\bar{3})(1,0,3)(\bar{1},\bar{0},\bar{3})2\overline{4132}4 = (1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2})2\underline{\overline{4132}}4$
$= (1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2})2(4,2,1,0,3)(\bar{4},\bar{2},\bar{1},\bar{0},\bar{3})31404$
$= (1,4)(0,2)(\bar{1},\bar{4})(\bar{0},\bar{2})131404 = (1,4)(0,2)(\bar{1},\bar{4})(\bar{0},\bar{2})\underline{1\bar{3}}31404$
$= (1,4)(0,2)(\bar{1},\bar{4})(\bar{0},\bar{2})(2,4,0)(\bar{2},\bar{4},\bar{0})3\overline{13}1404 = (1,0,4)(\bar{1},\bar{0},\bar{4})31\underline{1\bar{3}}1404$
$= (1,0,4)(\bar{1},\bar{0},\bar{4})31(2,4,0)(\bar{2},\bar{4},\bar{0})3\bar{1}1404 = (1,2,4)(\bar{1},\bar{2},\bar{4})313404.$ □

Since, $N^{(010232)} \geq N^{010232} = < e >$, and
$N(t_0t_1t_0t_2t_3t_2)^{\{(1,\bar{1})(2,\bar{3})(3,\bar{2})(4,\bar{4})(0,\bar{0})\}} = \overline{010323} \Rightarrow (1,\bar{1})(2,\bar{3})(3,\bar{2})(4,\bar{4})(0,\bar{0}) \in N^{(010232)}$
$N(t_0t_1t_0t_2t_3t_2)^{\{(2,4)(3,0)(\bar{2},\bar{4})(\bar{3},\bar{0})\}} = 313404 \Rightarrow (2,4)(3,0)(\bar{2},\bar{4})(\bar{3},\bar{0}) \in N^{(010232)}$
$\Rightarrow N^{(010232)} = < (1,\bar{1})(2,\bar{3})(3,\bar{2})(4,\bar{4})(0,\bar{0}), (2,4)(3,0)(\bar{2},\bar{4})(\bar{3},\bar{0}) >$
$= \{e, (1,\bar{1})(2,\bar{3})(3,\bar{2})(4,\bar{4})(0,\bar{0}), (2,4)(3,0)(\bar{2},\bar{4})(\bar{3},\bar{0}), (1,\bar{1})(2,\bar{4},3,\bar{0})(4,\bar{3},0,\bar{2}),$
$(2,0)(3,4)(\bar{2},\bar{0})(\bar{3},\bar{4}), (1,\bar{1})(2,\bar{0},3,\bar{4})(4,\bar{2},0,\bar{3}), (2,3)(4,0)(\bar{2},\bar{3})(\bar{4},\bar{0}),$
$(1,\bar{1})(2,\bar{2})(3,\bar{3})(4,\bar{0})(0,\bar{4})\}.$

Now, take (010232) conjugates with elements of $N^{(010232)}$, we obtain the following relations:

$$010232 \sim 212040 \sim 313404 \sim \overline{313040} \sim \overline{212404} \sim 414323 \sim \overline{010323} \sim \overline{414232}$$

Therefore, the number of single cosets in the double coset [010232] are at most

$\frac{|N|}{|N^{(010232)}|} = \frac{5!}{8} = \frac{120}{20} = 15$. Each single coset has eight different names.

By conjugating ($010232 \sim 212040 \sim 313404 \sim \overline{313040} \sim \overline{212404} \sim 414323 \sim \overline{010323} \sim \overline{414232}$) with the transversal of $N^{(010232)}$, we obtain the following fifteen single cosets of double coset [010232]:

1. $010232 \sim 212040 \sim 313404 \sim \overline{313040} \sim \overline{212404} \sim 414323 \sim \overline{010323} \sim \overline{414232}$
2. $121343 \sim 323101 \sim 424010 \sim \overline{424101} \sim \overline{323010} \sim 020434 \sim \overline{121434} \sim \overline{020343}$
3. $\overline{010242} \sim \overline{212030} \sim \overline{414303} \sim 414030 \sim 212303 \sim \overline{313424} \sim 010424 \sim 313242$
4. $232404 \sim 434212 \sim 030121 \sim \overline{030212} \sim \overline{434121} \sim 131040 \sim \overline{232040} \sim \overline{131404}$
5. $\overline{121303} \sim \overline{323141} \sim \overline{020414} \sim 020141 \sim 323414 \sim \overline{424030} \sim 121030 \sim 424303$
6. $343010 \sim 040323 \sim 141232 \sim \overline{141323} \sim \overline{040232} \sim 242101 \sim \overline{343101} \sim \overline{242010}$
7. $\overline{242303} \sim \overline{343212} \sim \overline{040121} \sim 040212 \sim 343121 \sim \overline{141030} \sim 242030 \sim 141303$

8. $\overline{434202} \sim \overline{232414} \sim \overline{030141} \sim 030414 \sim 232141 \sim \overline{131020} \sim 434020 \sim 131202$
9. $\overline{323404} \sim \overline{424313} \sim \overline{020131} \sim 020313 \sim 424131 \sim \overline{121040} \sim 323040 \sim 121404$
10. $404121 \sim 101434 \sim 202343 \sim \overline{202434} \sim \overline{101343} \sim 303212 \sim \overline{404212} \sim \overline{303121}$
11. $\overline{434010} \sim \overline{030424} \sim \overline{131242} \sim 131424 \sim 030242 \sim \overline{232101} \sim 434101 \sim 232010$
12. $\overline{303414} \sim \overline{404323} \sim \overline{101232} \sim 101323 \sim 404232 \sim \overline{202141} \sim 303141 \sim 202414$
13. $343202 \sim 242313 \sim 040131 \sim \overline{040313} \sim \overline{242131} \sim 141020 \sim \overline{343020} \sim \overline{141202}$
14. $010343 \sim 313020 \sim 414202 \sim \overline{303242} \sim \overline{101424} \sim 212434 \sim \overline{010434} \sim \overline{212343}$
15. $202131 \sim 101242 \sim 303424 \sim \overline{303242} \sim \overline{101424} \sim 404313 \sim \overline{202313} \sim \overline{404131}$

The orbits of $N^{(010232)}$ on $\{0, 1, 2, 3, 4, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ are $\{1, \bar{1}\}$ and $\{2, 3, 4, 0, \bar{2}, \bar{3}, \bar{4}, \bar{0}\}$. Now, take one representative from each orbit, namely 1 and $\bar{2}$, then find the double coset to which $Nt_0t_1t_0t_2t_3t_2t_1$ and $Nt_0t_1t_0t_2t_3t_2t_{\bar{2}}$ belong. However,

$Nt_0t_1t_0t_2t_3t_2t_1 = Nt_0t_1t_0t_2t_3t_2t_1 \in [010232]$**(By Lemma-86).**
$Nt_0t_1t_0t_2t_3t_2t_{\bar{2}} = Nt_0t_1t_0t_2t_3 \in [01023]$.

Therefore, all symmetric generators go back to previous double cosets and none of them extends. Thus, we must have completed the double coset enumeration. The double coset enumeration shows that the index of $N \cong S_5$ in G is at most

$$|G| \leq (1,584)|N| = 1,584(120) = 190,080.$$

The information are summarized in the Cayley diagram below.

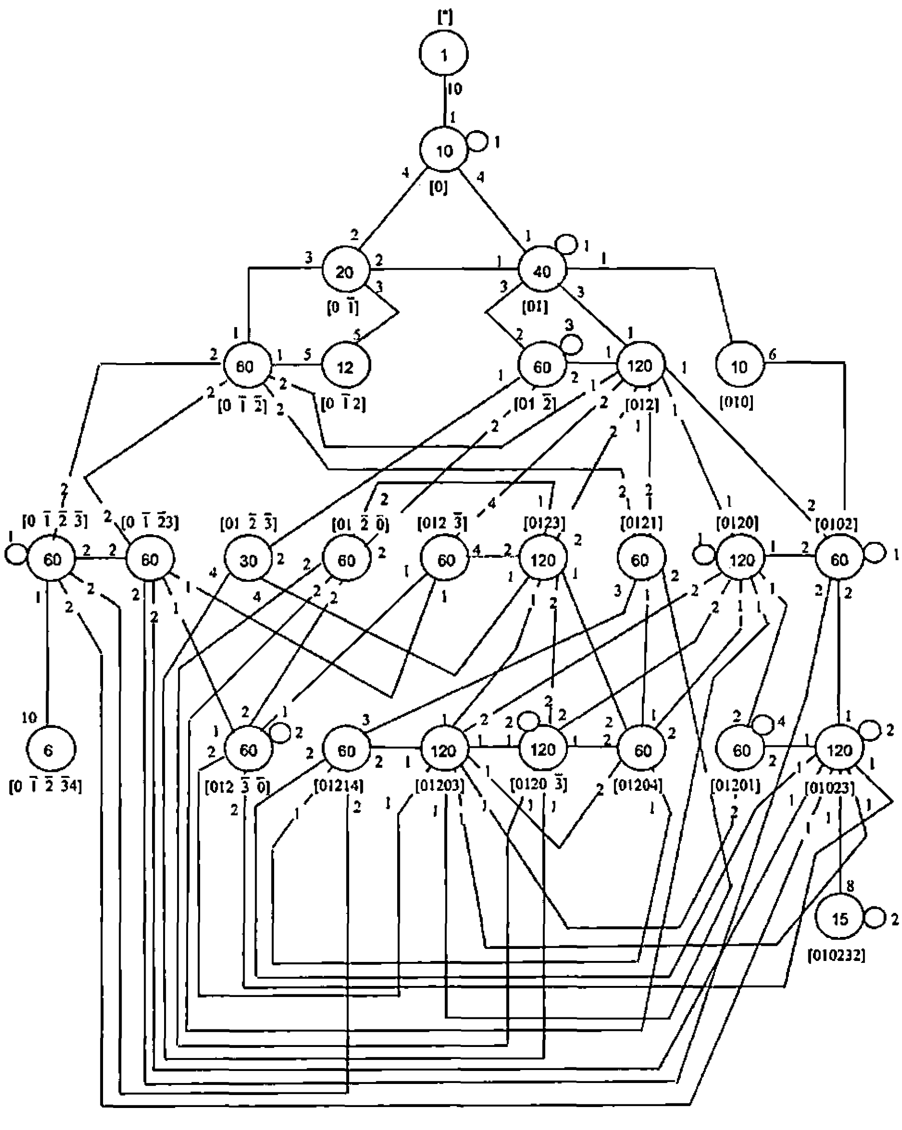

Figure 9.1: The Cayley diagram of $M_{12} \times 2$ over S_5.

The homomorphic image of G.

Define $\widehat{\alpha}: G \to S_{1584}$, where (1584) is a number of single cosets from the Cayley diagram above gives the action of G on the 1584 cosets. From the labeling table (too large to show), we compute the actions of x, y and t_0 on the 1584 cosets.

$\widehat{\alpha}(x) = (2,4,8,13,5)(3,6,15,11,7)(9,22,44,49,23)...(1566,1576,1583,1569,1570)$.

$\widehat{\alpha}(y) = (2,3)(4,6)(5,11)(7,13)(8,15)(9,17)...(1572,1579)(1578,1582)(1583,1584)$.

$\widehat{\alpha}(t_0) = (1,2,3)(4,9,10)(5,12,14)(6,16,17)...(1522,1534,1562)(1557,1583,1584)$.

We now want to verify that $\widehat{\alpha}$ is a homomorphism.

1. We want to show that $\widehat{\alpha}(t_0)$ has exactly 10 conjugates under conjugation by $<\widehat{\alpha}(x), \widehat{\alpha}(y)>$.

   $\widehat{\alpha}(t_0)^{\widehat{\alpha}(x)} = (1,4,6)(2,27,10)(3,16,38)...(1547,1577,1581) = \widehat{\alpha}(t_1)$.

   $\widehat{\alpha}(t_1)^{\widehat{\alpha}(x)} = (1,8,15)(2,129,21)(3,32,114)...(1431,1550,1570) = \widehat{\alpha}(t_2)$.

   $\widehat{\alpha}(t_2)^{\widehat{\alpha}(x)} = (1,13,11)(2,77,30)(3,25,47)...(1404,1506,1547) = \widehat{\alpha}(t_3)$.

   $\widehat{\alpha}(t_3)^{\widehat{\alpha}(x)} = (1,5,7)(2,23,14)(3,18,36)...(1546,1572,1582) = \widehat{\alpha}(t_4)$.

   $\widehat{\alpha}(t_4)^{\widehat{\alpha}(x)} = (1,2,3)(4,9,10)(5,12,14)...(1557,1583,1584) = \widehat{\alpha}(t_0)$.

   $\widehat{\alpha}(\bar{t_0})^{\widehat{\alpha}(x)} = (1,6,4)(2,10,27)(3,38,16)...(1547,1581,1577) = \widehat{\alpha}(\bar{t_1})$.

   $\widehat{\alpha}(\bar{t_1})^{\widehat{\alpha}(x)} = (1,15,8)(2,21,129)(3,114,32)...(1431,1570,1550) = \widehat{\alpha}(\bar{t_2})$.

   $\widehat{\alpha}(\bar{t_2})^{\widehat{\alpha}(x)} = (1,11,13)(2,30,77)(3,47,25)...(1404,1547,1506) = \widehat{\alpha}(\bar{t_3})$.

   $\widehat{\alpha}(\bar{t_3})^{\widehat{\alpha}(x)} = (1,7,5)(2,14,23)(3,36,18)...(1546,1582,1572) = \widehat{\alpha}(\bar{t_4})$.

   $\widehat{\alpha}(\bar{t_4})^{\widehat{\alpha}(x)} = (1,3,2)(4,10,9)(5,14,12)...(1557,1584,1583) = \widehat{\alpha}(\bar{t_0})$.

   $\widehat{\alpha}(t_0)^{\widehat{\alpha}(y)} = (1,3,2)(4,10,9)(5,14,12)...(1557,1584,1583) = \widehat{\alpha}(\bar{t_0})$.

   $\widehat{\alpha}(\bar{t_0})^{\widehat{\alpha}(y)} = (1,2,3)(4,9,10)(5,12,14)...(1557,1583,1584) = \widehat{\alpha}(t_0)$.

   $\widehat{\alpha}(t_1)^{\widehat{\alpha}(y)} = (1,6,4)(2,10,27)(3,38,16)...(1547,1581,1577) = \widehat{\alpha}(\bar{t_1})$.

   $\widehat{\alpha}(\bar{t_1})^{\widehat{\alpha}(y)} = (1,4,6)(2,27,10)(3,16,38)...(1547,1577,1581) = \widehat{\alpha}(t_1)$.

   $\widehat{\alpha}(t_2)^{\widehat{\alpha}(y)} = (1,15,8)(2,21,129)(3,114,32)...(1431,1570,1550) = \widehat{\alpha}(\bar{t_2})$.

   $\widehat{\alpha}(\bar{t_2})^{\widehat{\alpha}(y)} = (1,8,15)(2,129,21)(3,32,114)...(1431,1550,1570) = \widehat{\alpha}(t_2)$.

   $\widehat{\alpha}(t_3)^{\widehat{\alpha}(y)} = (1,7,5)(2,14,23)(3,36,18)...(1546,1582,1572) = \widehat{\alpha}(\bar{t_4})$.

   $\widehat{\alpha}(\bar{t_4})^{\widehat{\alpha}(y)} = (1,13,11)(2,77,30)(3,25,47)...(1404,1506,1547) = \widehat{\alpha}(t_3)$.

   $\widehat{\alpha}(\bar{t_3})^{\widehat{\alpha}(y)} = (1,5,7)(2,23,14)(3,18,36)...(1546,1572,1582) = \widehat{\alpha}(t_4)$.

$\widehat{\alpha}(t_4)^{\widehat{\alpha}(y)} = (1,11,13)(2,30,77)(3,47,25)...(1404,1547,1506) = \widehat{\alpha}(\bar{t_3})$.

2. $\widehat{\alpha}(N)$ acts as N on $\{\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(t_2), \widehat{\alpha}(t_3), \widehat{\alpha}(t_4), \widehat{\alpha}(\bar{t_0}), \widehat{\alpha}(\bar{t_1}), \widehat{\alpha}(\bar{t_2}), \widehat{\alpha}(\bar{t_3}), \widehat{\alpha}(\bar{t_4})\}$ by conjugation. We note that
$\widehat{\alpha}(x) = \{(\widehat{\alpha}(t_0), \widehat{\alpha}(t_1), \widehat{\alpha}(t_2), \widehat{\alpha}(t_3), \widehat{\alpha}(t_4))(\widehat{\alpha}(\bar{t_0}), \widehat{\alpha}(\bar{t_1}), \widehat{\alpha}(\bar{t_2}), \widehat{\alpha}(\bar{t_3}), \widehat{\alpha}(\bar{t_4}))\}$,
$\widehat{\alpha}(y) = \{(\widehat{\alpha}(t_0), \widehat{\alpha}(\bar{t_0}))(\widehat{\alpha}(t_1), \widehat{\alpha}(\bar{t_1}))(\widehat{\alpha}(t_2), \widehat{\alpha}(\bar{t_2}))(\widehat{\alpha}(t_3), \widehat{\alpha}(\bar{t_4}))(\widehat{\alpha}(t_4), \widehat{\alpha}(\bar{t_3}))\}$.
And, $< \widehat{\alpha}(x), \widehat{\alpha}(x) > \cong N = S_5$.
Then, $\widehat{\alpha}(G) = \widehat{\alpha}(3^{*5} : S_5) = \widehat{\alpha} < N, t_0 > = < \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$ is a homomorphic image of $(3^{*5} : S_5)$.
Next, we want the homomorphic image of
$G = \frac{3^{*5} :_m S_5}{[(0,1)(\bar{0},\bar{1})(2,\bar{3},4,\bar{2},3,\bar{4})t_0]^4 [(0,1,2,3,4)(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4})t_0]^8}$.
Now, $< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) > \leq S_{1584}$ is a homomorphic image of
$G = \frac{3^{*5} :_m S_5}{[(0,1)(\bar{0},\bar{1})(2,\bar{3},4,\bar{2},3,\bar{4})t_0]^4 [(0,1,2,3,4)(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4})t_0]^8}$ if the additional relation
$(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{t_1}t_0 = \bar{t_0}t_1$ and $(0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})t_2t_1t_0t_4 = \bar{t_0}\bar{t_1}\bar{t_2}\bar{t_3}$ hold in $< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$.

Relation-I

We want to show that $\widehat{\alpha}((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{t_1}t_0) = \widehat{\alpha}(\bar{t_0}t_1)$.

On the left side of our equation

$\widehat{\alpha}((2,3,4)(\bar{2},\bar{3},\bar{4})\bar{t_1}t_0) = \widehat{\alpha}((2,3,4)(\bar{2},\bar{3},\bar{4}))\widehat{\alpha}(\bar{t_1})\widehat{\alpha}(t_0)$

$= (5,8,13)(7,15,11)(12,20,29)(14,21,30)...(1558,1576,1570)(1578,1582,1580)$

$(1,6,4)(2,10,27)(3,38,16)(5,43,63)(7,55,53)...(1511,1582,1553)(1547,1581,1577)$

$(1,2,3)(4,9,10)(5,12,14)(6,16,17)(7,18,19)...(1522,1534,1562)(1557,1583,1584)$

$= (1,16)(2,4)(3,27)(5,51,8,90,13,92)...(1443,1574,1573)(1547,1577,1581)$.

On the right side of our equation

$\widehat{\alpha}(\bar{t_0}t_1) = \widehat{\alpha}(\bar{t_0})\widehat{\alpha}(t_1)$

$= (1,3,2)(4,10,9)(5,14,12)(6,17,16)...(1522,1562,1534)(1557,1584,1583)$

$(1,4,6)(2,27,10)(3,16,38)(5,63,43)...(1511,1553,1582)(1547,1577,1581)$

$= (1,16)(2,4)(3,27)(5,51,8,90,13,92)...(1443,1574,1573)(1547,1577,1581)$.

Relation-II

We want to show that $\widehat{\alpha}((0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})t_2t_1t_0t_4) = \widehat{\alpha}(\bar{t_0}\bar{t_1}\bar{t_2}\bar{t_3})$.

On the left side of our equation

$\widehat{\alpha}((0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})t_2t_1t_0t_4)$

$= \widehat{\alpha}((0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2}))\widehat{\alpha}(t_2)\widehat{\alpha}(t_1)\widehat{\alpha}(t_0)\widehat{\alpha}(t_4)$

$= (2,13,4,5,8)(3,11,6,7,15)(9,49,22,23,44)...(1566,1569,1576,1570,1583)$

$(1,8,15)(2,129,21)(3,32,114)(4,57,24)...(1418,1568,1580)(1431,1550,1570)$

$(1,4,6)(2,27,10)(3,16,38)(5,63,43)...(1511,1553,1582)(1547,1577,1581)$

$(1,2,3)(4,9,10)(5,12,14)(6,16,17)...(1522,1534,1562)(1557,1583,1584)$

$(1,5,7)(2,23,14)(3,18,36)(4,41,43)...(1508,1583,1565)(1546,1572,1582)$

$= (1,346,67,320,183)(2,299,240,111,443)...(1024,1542,1115,1514,1099).$

On the right side of our equation

$\widehat{\alpha}(\bar{t_0}\bar{t_1}\bar{t_2}\bar{t_3})$

$= (1,3,2)(4,10,9)(5,14,12)(6,17,16)...(1522,1562,1534)(1557,1584,1583)$

$(1,6,4)(2,10,27)(3,38,16)(5,43,63)...(1511,1582,1553)(1547,1581,1577)$

$(1,15,8)(2,21,129)(3,114,32)(4,24,57)...(1418,1580,1568)(1431,1570,1550)$

$(1,11,13)(2,30,77)(3,47,25)(4,42,133)...(1385,1473,1522)(1404,1547,1506)$

$= (1,346,67,320,183)(2,299,240,111,443)...(1024,1542,1115,1514,1099).$

Therefore, $\widehat{\alpha} : G \to S_{1584}$ is a homomorphism map.

Now by First Isomorphism Theorem, we have

$G|ker(\widehat{\alpha}) \cong im(\widehat{\alpha})$

$\Rightarrow \frac{|G|}{|ker(\widehat{\alpha})|} = |im(\widehat{\alpha})| = | < \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0)| = 190,080$

$\Rightarrow |G| = 190,080 \times |ker(\widehat{\alpha})|$, where $(|ker(\widehat{\alpha})| \geq 1)$.

$\Rightarrow |G| \geq 190,080$.

But early we found that $|G| \leq 190,080$. Therefore, we can conclude that

$|G| = 190,080$. Since $|G| = 190,080$,

$\Rightarrow |ker(\widehat{\alpha})| = 1$.

$\Rightarrow \widehat{\alpha}$ is also an isomorphism.

$\Rightarrow G \cong \widehat{\alpha}(G) =< \widehat{\alpha}(x), \widehat{\alpha}(y), \widehat{\alpha}(t_0) >$.

It is clear from the Cayley diagram of G over S_5 that every element of G can be written as πw, where π is a permutation of S_5 on ten letters, and w is a word in the t_{i_s} of the length at most six. We call this representation of elements of G the symmetric representation. We now give examples of elements of G in both symmetric and permutation representation.

Permutation Representation to Symmetric Presentation

Let $p = (1, 35, 144, 766, 995, 644, 92, 40)(2, 108, 21, 388, 597, 917, 141, 177)$
$(3, 150, 96, 999, 564, 994, 18, 261)(4, 6, 24, 165, 570, 898, 329, 68)...$
$(962, 1090, 982, 1143, 1264)(1024, 1542, 1115, 1514, 1099)$
be an element in Permutation Representation.
We know that $Np = 1^p$
$\Rightarrow Np = p(1) = 35$ and the table gives that $N\bar{t_2}\bar{t_1}$ is labeled 35.
In order the represent p in the symmetric representation form, we must find $n \in N \cong S_5$.
$n = pw^{-1}$.
$Np = N\bar{t_2}\bar{t_1}$. (An element that labeling (35)).
$p = nw$
$\Rightarrow n = pw^{-1}$
$= p(\bar{t_2}\bar{t_1})^{-1}$
$\Rightarrow \widehat{\alpha}(n) = p\widehat{\alpha}(t_1)\widehat{\alpha}(t_2)$
$= (2, 13, 4, 8, 5)(3, 11, 6, 15, 7)(9, 61, 63, 129, 28)(10, 50, 43, 21, 31)...$
$(1408, 1534, 1435, 1581, 1442)(1431, 1560, 1583, 1558, 1569)$.
From the labeling table we have,
$n = (0, 3, 1, 2, 4)(\bar{0}, \bar{3}, \bar{1}, \bar{2}, \bar{4})$.
Therefore, p can be written in the symmetric representation as
$p = (0, 3, 1, 2, 4)(\bar{0}, \bar{3}, \bar{1}, \bar{2}, \bar{4})\bar{t_2}\bar{t_1}$.

Symmetric Presentation to Permutation Representation

Let $\{(2, 3, 4)(\bar{2}, \bar{3}, \bar{4})\bar{t_1}t_0\}$ be an element in the symmetric representation.
$\Rightarrow p = \widehat{\alpha}((2, 3, 4)(\bar{2}, \bar{3}, \bar{4}))\widehat{\alpha}(\bar{t_1})\widehat{\alpha}(t_0)$
$= (1, 16)(2, 4)(3, 27)(5, 90)(6, 9)(7, 139)(8, 92)(10, 58)(11, 75)(12, 22)(13, 51)(14, 168)$
$(15, 115)(17, 38)(18, 34)(19, 210)(20, 40)(21, 244)(23, 215)(24, 130)(25, 53)(26, 152)$
$(28, 156)(29, 63)(30, 124)(31, 205)(32, 68)(33, 212)(35, 110)(36, 561)(39, 506)(41, 166)$
$(42, 45)(43, 85)(44, 273)(46, 316)(47, 364)...(1451, 1478)(1462, 1570)(1469, 1534)$
$(1474, 1580)(1487, 1500)(1499, 1551)(1509, 1564)(1510, 1558)(1511, 1578)(1513, 1527)$
$(1525, 1548)(1552, 1557)(1553, 1571)(1555, 1583)$.

Therefore, $\{(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{t_2}t_4\}$ can be written in the permutation representation as follows

$(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{t_2}t_4$
$= (1,16)(2,4)(3,27)(5,90)(6,9)(7,139)(8,92)(10,58)(11,75)(12,22)(13,51)(14,168)$
$(15,115)(17,38)(18,34)(19,210)(20,40)(21,244)(23,215)(24,130)(25,53)(26,152)$
$(28,156)(29,63)(30,124)(31,205)(32,68)(33,212)(35,110)(36,561)(39,506)(41,166)$
$(42,45)(43,85)(44,273)(46,316)(47,364)...(1451,1478)(1462,1570)(1469,1534)$
$(1474,1580)(1487,1500)(1499,1551)(1509,1564)(1510,1558)(1511,1578)(1513,1527)$
$(1525,1548)(1552,1557)(1553,1571)(1555,1583).$

9.2 Lemmas

Lemma-1 $(01\bar{0} \in [01])$

Proof. $0\underline{1\bar{0}} = 0(2,3,4)(\bar{2},\bar{3},\bar{4})0\bar{1} = (2,3,4)(\bar{2},\bar{3},\bar{4})00\bar{1} = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1} \in [01].$ □

Lemma-2 $(0124 \in [0123])$

Proof. $\underline{0124} = (0,4,1,3,2)(\bar{0},\bar{4},\bar{1},\bar{3},\bar{2})\bar{2}\bar{1}\bar{0}\bar{3} \in [0123].$ □

Lemma-3 $(012\bar{0} \in 0102)$

Proof. $012\bar{0} = \bar{0}\underline{01}2\bar{0} = \bar{0}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}02\bar{0} = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}$
$02\bar{0} = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}\bar{0}\underline{02}\bar{0} = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}\bar{0}(1,4,3)(\bar{1},\bar{4},\bar{3})\bar{2}0\bar{0} = (2,1,4)(\bar{2},\bar{1},\bar{4})$
$\bar{0}\bar{4}\bar{0}\bar{2} \in [0102].$ □

Lemma-4 $(012\bar{1} \in [0\bar{1}\bar{2}])$

Proof. $01\underline{2\bar{1}} = 01(0,3,4)(\bar{0},\bar{3},\bar{4})1\bar{2} = (0,3,4)(\bar{0},\bar{3},\bar{4})311\bar{2}$
$= (0,3,4)(\bar{0},\bar{3},\bar{4})3\bar{1}\bar{2} \in [0\bar{1}\bar{2}].$ □

Lemma-5 $(012\bar{4} \in [012\bar{3}])$

Proof. $012\bar{4} = \underline{0124}4 = (0,4,1,3,2)(\bar{0},\bar{4},\bar{1},\bar{3},\bar{2})\bar{2}\bar{1}\bar{0}\underline{34}$
$= (0,4,1,3,2)(\bar{0},\bar{4},\bar{1},\bar{3},\bar{2})\bar{2}\bar{1}\bar{0}(2,0,1)(\bar{2},\bar{0},\bar{1})\bar{4}3 = (0,4,2,1,3)(\bar{0},\bar{4},\bar{2},\bar{1},\bar{3})\underline{\bar{0}\bar{2}\bar{1}\bar{4}}3$
$= (0,4,2,1,3)(\bar{0},\bar{4},\bar{2},\bar{1},\bar{3})(0,4,2,3,1)(\bar{0},\bar{4},\bar{2},\bar{3},\bar{1})12033$
$= (0,2)(4,3)(\bar{0},\bar{2})(\bar{4},\bar{3})120\bar{3} \in 012\bar{3}.$ □

Lemma-6 ($01\bar{2}3 \in [01\bar{2}]$)

Proof. $01\bar{2}3 = \bar{0}\underline{\bar{0}1}\bar{2}3 = \bar{0}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0\bar{2}3 = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}\underline{0\bar{2}}3$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}(1,3,4)(\bar{1},\bar{3},\bar{4})2\bar{0}3 = (2,4)(3,1)(\bar{2},\bar{4})(\bar{3},\bar{1})\bar{0}\bar{3}2\underline{\bar{0}3}$
$= (2,4)(3,1)(\bar{2},\bar{4})(\bar{3},\bar{1})\bar{0}\bar{3}2(1,2,4)(\bar{1},\bar{2},\bar{4})\bar{3}0 = (2,1,3)(\bar{2},\bar{1},\bar{3})\bar{0}\bar{3}\underline{4\bar{3}}0$
$= (2,1,3)(\bar{2},\bar{1},\bar{3})\bar{0}\bar{3}(2,0,1)(\bar{2},\bar{0},\bar{1})3\bar{4}0 = (1,3,0)(\bar{1},\bar{3},\bar{0})\bar{1}33\bar{4}0$
$= (1,3,0)(\bar{1},\bar{3},\bar{0})\bar{1}\bar{4}0 \in [01\bar{2}]$. □

Lemma-7 ($01\bar{2}0 \in [01\bar{2}]$)

Proof. $01\bar{2}0 = \bar{0}\underline{\bar{0}1}\bar{2}0 = \bar{0}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0\bar{2}0 = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}\underline{0\bar{2}}0$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}(1,3,4)(\bar{1},\bar{3},\bar{4})2\bar{0}0 = (2,4)(3,1)(\bar{2},\bar{4})(\bar{3},\bar{1})\bar{0}\bar{3}2 \in [01\bar{2}]$. □

Lemma-8 ($0\bar{1}\bar{2}0 \in [012]$)

Proof. $0\bar{1}\bar{2}0 = (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}\bar{2}0 = (2,4,3)(\bar{2},\bar{4},\bar{3})10\underline{0\bar{2}}0$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})10(1,3,4)(\bar{1},\bar{3},\bar{4})2\bar{0}0 = (2,1,3)(\bar{2},\bar{1},\bar{3})302 \in [012]$. □

Lemma-9 ($0\bar{1}\bar{2}\bar{0} \in [0121]$)

Proof. $\underline{0\bar{1}}\bar{2}\bar{0} = (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}\bar{2}\bar{0} = (2,4,3)(\bar{2},\bar{4},\bar{3})10\underline{0\bar{2}}\bar{0}$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})10(1,3,4)(\bar{1},\bar{3},\bar{4})2\bar{0}\bar{0} = (2,1,3)(\bar{2},\bar{1},\bar{3})3020 \in [0121]$. □

Lemma-10 ($010\bar{2} \in [0102]$)

Proof. $01\underline{0\bar{2}} = 01(1,3,4)(\bar{1},\bar{3},\bar{4})2\bar{0} = (1,3,4)(\bar{1},\bar{3},\bar{4})032\bar{0} = (1,3,4)(\bar{1},\bar{3},\bar{4})\bar{0}\underline{\bar{0}3}2\bar{0}$
$= (1,3,4)(\bar{1},\bar{3},\bar{4})\bar{0}(1,2,4)(\bar{1},\bar{2},\bar{4})2\bar{0} = (1,3)(4,2)(\bar{1},\bar{3})(\bar{4},\bar{2})\bar{0}\bar{3}02\bar{0}$
$= (1,3)(4,2)(\bar{1},\bar{3})(\bar{4},\bar{2})0\underline{0\bar{3}}02\bar{0} = (1,3)(4,2)(\bar{1},\bar{3})(\bar{4},\bar{2})0(1,4,2)(\bar{1},\bar{4},\bar{2})3\bar{0}02\bar{0}$
$= (1,3,4)(\bar{1},\bar{3},\bar{4})03\underline{2\bar{0}} = (1,3,4)(\bar{1},\bar{3},\bar{4})03(1,3,4)(\bar{1},\bar{3},\bar{4})0\bar{2}$
$= (1,4,3)(\bar{1},\bar{4},\bar{3})040\bar{2} = (1,4,3)(\bar{1},\bar{4},\bar{3})\bar{0}\underline{\bar{0}4}0\bar{2}$
$= (1,4,3)(\bar{1},\bar{4},\bar{3})\bar{0}(2,1,3)(\bar{2},\bar{1},\bar{3})\bar{4}00\bar{2} = (1,4,2)(\bar{1},\bar{4},\bar{2})\bar{0}\bar{4}\bar{0}\bar{2} \in [0102]$. □

Lemma-11 ($01020 \in [0120]$)

Proof. $01020 = \bar{0}\underline{\bar{0}1}020 = \bar{0}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0020 = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}\underline{02}0$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}(1,4,3)(\bar{1},\bar{4},\bar{3})\bar{2}00 = (2,1,4)(\bar{2},\bar{1},\bar{4})\bar{0}\bar{4}\bar{2}\bar{0} \in [0120]$. □

Lemma-12 ($0102\bar{0} \in [012]$)

Proof. $010\underline{2\bar{0}} = 010(1,4,3)(\bar{1},\bar{4},\bar{3})0\bar{2} = (1,4,3)(\bar{1},\bar{4},\bar{3})0400\bar{2} = (1,4,3)(\bar{1},\bar{4},\bar{3})0\underline{4\bar{0}}\bar{2}$
$= (1,4,3)(\bar{1},\bar{4},\bar{3})0(1,3,2)(\bar{1},\bar{3},\bar{2})0\bar{4}\bar{2} = (1,4,3)(\bar{1},\bar{4},\bar{3})00\bar{4}\bar{2}$
$= (1,4,3)(\bar{1},\bar{4},\bar{3})\bar{0}\bar{4}\bar{2} \in [012].$ □

Lemma-13 ($0102\bar{3} \in [0\bar{1}\bar{2}3]$)

Proof. $0102\bar{3} = \bar{0}\underline{\bar{0}1}20\bar{2} = \bar{0}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}002\bar{3} = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}\underline{\bar{0}2}\bar{3}$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}(1,4,3)(\bar{1},\bar{4},\bar{3})\bar{2}0\bar{3} = (2,1,4)(\bar{2},\bar{1},\bar{4})\bar{0}\bar{4}\bar{2}\underline{0\bar{3}}$
$= (2,1,4)(\bar{2},\bar{1},\bar{4})\bar{0}\bar{4}\bar{2}(1,4,2)(\bar{1},\bar{4},\bar{2})3\bar{0} = (2,4,1)(\bar{2},\bar{4},\bar{1})\bar{0}\bar{2}\underline{\bar{1}3}\bar{0}$
$= (2,4,1)(\bar{2},\bar{4},\bar{1})\bar{0}\bar{2}(2,0,4)(\bar{2},\bar{0},\bar{4})\bar{3}1\bar{0} = (4,1,0)(\bar{4},\bar{1},\bar{0})\bar{4}\bar{0}\bar{3}\underline{1\bar{0}}$
$= (4,1,0)(\bar{4},\bar{1},\bar{0})\bar{4}\bar{0}\bar{3}(2,3,4)(\bar{2},\bar{3},\bar{4})0\bar{1} = (4,1,0,2,3)(\bar{4},\bar{1},\bar{0},\bar{2},\bar{3})\bar{2}\bar{0}\bar{4}0\bar{1}$
$= (4,1,0,2,3)(\bar{4},\bar{1},\bar{0},\bar{2},\bar{3})\bar{2}0\underline{0\bar{4}}0\bar{1} = (4,1,0,2,3)(\bar{4},\bar{1},\bar{0},\bar{2},\bar{3})\bar{2}0(1,2,3)(\bar{1},\bar{2},\bar{3})4\bar{0}0\bar{1}$
$= (4,2,1,0,3)(\bar{4},\bar{2},\bar{1},\bar{0},\bar{3})\bar{3}04\bar{1} \in [0\bar{1}\bar{2}3].$ □

Lemma-14 ($012\bar{0} \in [0102]$)

Proof. $012\bar{0} = \bar{0}\underline{\bar{0}1}2\bar{0} = \bar{0}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}02\bar{0} = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}0\underline{2\bar{0}}$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}0(1,4,3)(\bar{1},\bar{4},\bar{3})0\bar{2} = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{4}00\bar{2}$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{4}\bar{0}\bar{2} \in [0102].$ □

Lemma-15 ($01202 \in [0120]$)

Proof. $01202 = \bar{0}\underline{\bar{0}1}202 = \bar{0}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0202 = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}0202$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}\underline{\bar{0}2}02 = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}\bar{0}(1,4,3)(\bar{1},\bar{4},\bar{3})\bar{2}002$
$= (2,1,4)(\bar{2},\bar{1},\bar{4})\bar{0}\bar{4}\bar{0}\bar{2}\bar{0}2 = (2,1,4)(\bar{2},\bar{1},\bar{4})\bar{0}\bar{4}\bar{0}2\underline{2\bar{0}}2$
$= (2,1,4)(\bar{2},\bar{1},\bar{4})\bar{0}\bar{4}\bar{0}2(1,3,4)(\bar{1},\bar{3},\bar{4})0\bar{2}2 = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}\underline{\bar{0}2}0$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}(1,4,3)(\bar{1},\bar{4},\bar{3})\bar{2}00 = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{4}\bar{2}\bar{0} \in [0120].$ □

Lemma-16 ($0120\bar{1} \in [01203]$)

Proof. $012\underline{0\bar{1}} = 012(2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0} = (2,4,3)(\bar{2},\bar{4},\bar{3})0141\bar{0} = (2,4,3)(\bar{2},\bar{4},\bar{3})01\underline{4\bar{1}}\bar{1}\bar{0}$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})01(2,3,0)(\bar{2},\bar{3},\bar{0})1\bar{4}\bar{1}\bar{0} = (2,4,0)(\bar{2},\bar{4},\bar{0})211\bar{4}\bar{1}\bar{0}$
$= (2,4,0)(\bar{2},\bar{4},\bar{0})\underline{2\bar{1}}\bar{4}\bar{1}\bar{0} = (2,4,0)(\bar{2},\bar{4},\bar{0})(0,3,4)(\bar{0},\bar{3},\bar{4})1\bar{2}\bar{4}\bar{1}\bar{0}$
$= (2,0)(4,3)(\bar{2},\bar{0})(\bar{4},\bar{3})1\underline{\bar{2}\bar{4}\bar{1}\bar{0}} = (2,0)(4,3)(\bar{2},\bar{0})(\bar{4},\bar{3})1(2,0,4,3,1)(\bar{2},\bar{0},\bar{4},\bar{3},\bar{1})1423$
$= (2,4,1)(\bar{2},\bar{4},\bar{1})21423 \in [01203].$ □

Lemma-17 ($0120\bar{2} \in [01\bar{2}\bar{0}]$)

Proof. $0120\bar{2} = \bar{0}\underline{\bar{0}1}20\bar{2} = \bar{0}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}020\bar{2} = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}020\bar{2}$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}0\bar{2}\underline{\bar{2}0}\bar{2} = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}0\bar{2}(1,3,4)(\bar{1},\bar{3},\bar{4})\bar{0}2\bar{2}$
$= (2,4)(3,1)(\bar{2},\bar{4})(\bar{3},\bar{1})\bar{0}\bar{3}0\bar{2}\bar{0} = (2,4)(3,1)(\bar{2},\bar{4})(\bar{3},\bar{1})0\underline{0\bar{3}}0\bar{2}\bar{0}$
$= (2,4)(3,1)(\bar{2},\bar{4})(\bar{3},\bar{1})0(1,4,2)(\bar{1},\bar{4},\bar{2})3\bar{0}0\bar{2}\bar{0} = (4,1,3)(\bar{4},\bar{1},\bar{3})03\bar{2}\bar{0} \in [01\bar{2}\bar{0}]$. □

Lemma-18 ($0120\bar{4} \in [0120\bar{3}]$)

Proof. $0120\bar{4} = 0\underline{1204}4 = 0(1,4,2,3,0)(\bar{1},\bar{4},\bar{2},\bar{3},\bar{0})\bar{0}\bar{2}\bar{1}\bar{3}4$
$= (1,4,2,3,0)(\bar{1},\bar{4},\bar{2},\bar{3},\bar{0})1\bar{0}\bar{2}\bar{1}\underline{\bar{3}4} = (1,4,2,3,0)(\bar{1},\bar{4},\bar{2},\bar{3},\bar{0})1\bar{0}\bar{2}\bar{1}(2,0,1)(\bar{2},\bar{0},\bar{1})\bar{4}3$
$= (1,4,0,2,3)(\bar{1},\bar{4},\bar{0},\bar{2},\bar{3})2\underline{\bar{1}\bar{0}\bar{2}\bar{4}}3$
$= (1,4,0,2,3)(\bar{1},\bar{4},\bar{0},\bar{2},\bar{3})2(1,4,0,3,2)(\bar{1},\bar{4},\bar{0},\bar{3},\bar{2})20133$
$= (1,0)(4,3)(\bar{1},\bar{0})(\bar{4},\bar{3})1201\bar{3} \in [0120\bar{3}]$. □

Lemma-19 ($0121\bar{4} \in [01204]$)

Proof. $0121\bar{4} = 01\underline{2\bar{1}}\bar{1}\bar{4} = 01(0,3,4)(\bar{0},\bar{3},\bar{4})1\bar{2}\bar{1}\bar{4} = (0,3,4)(\bar{0},\bar{3},\bar{4})311\bar{2}\bar{1}\bar{4}$
$= (0,3,4)(\bar{0},\bar{3},\bar{4})\underline{3\bar{1}}\bar{2}\bar{1}\bar{4} = (0,3,4)(\bar{0},\bar{3},\bar{4})(2,0,4)(\bar{2},\bar{0},\bar{4})1\bar{3}\bar{2}\bar{1}\bar{4}$
$= (0,3,2)(\bar{0},\bar{3},\bar{2})1\underline{\bar{3}\bar{2}\bar{1}\bar{4}} = (0,3,2)(\bar{0},\bar{3},\bar{2})1(1,3,4,2,0)(\bar{1},\bar{3},\bar{4},\bar{2},\bar{0})1230$
$= (0,4,2,1,3)(\bar{0},\bar{4},\bar{2},\bar{1},\bar{3})31230 \in [01204]$. □

Lemma-20 ($012\bar{1} \in [0\bar{1}\bar{2}]$)

Proof. $012\bar{1} = 0\bar{1}\underline{\bar{1}2}\bar{1} = 0\bar{1}(0,3,4)(\bar{0},\bar{3},\bar{4})\bar{2}1\bar{1} = (0,3,4)(\bar{0},\bar{3},\bar{4})3\bar{1}\bar{2} \in [0\bar{1}\bar{2}]$. □

Lemma-21 ($01210 \in [01214]$)

Proof. $01210 = 01\underline{2\bar{1}}\bar{1}0 = 01(0,3,4)(\bar{0},\bar{3},\bar{4})1\bar{2}\bar{1}0 = (0,3,4)(\bar{0},\bar{3},\bar{4})311\bar{2}\bar{1}0$
$= (0,3,4)(\bar{0},\bar{3},\bar{4})\underline{3\bar{1}}\bar{2}\bar{1}0 = (0,3,4)(\bar{0},\bar{3},\bar{4})(2,0,4)(\bar{2},\bar{0},\bar{4})1\bar{3}\bar{2}\bar{1}0$
$= (0,3,2)(\bar{0},\bar{3},\bar{2})1\bar{3}\bar{2}\bar{1}0 = (0,3,2)(\bar{0},\bar{3},\bar{2})1\underline{\bar{3}\bar{2}\bar{1}0}\bar{0}$
$= (0,3,2)(\bar{0},\bar{3},\bar{2})1(3,0,2,1,4)(\bar{3},\bar{0},\bar{2},\bar{1},\bar{4})1234\bar{0} = (3,4,1)(\bar{3},\bar{4},\bar{1})3123\underline{4\bar{0}}$
$= (3,4,1)(\bar{3},\bar{4},\bar{1})3123(1,3,2)(\bar{1},\bar{3},\bar{2})0\bar{4} = (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})2\underline{3120}\bar{4}$
$= (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})2(3,0,1,4,2)(\bar{3},\bar{0},\bar{1},\bar{4},\bar{2})\bar{2}\bar{1}\bar{3}\bar{4}\bar{4}$
$= (3,2,4,0,1)(\bar{3},\bar{2},\bar{4},\bar{0},\bar{1})\underline{3\bar{2}}\bar{1}\bar{3}4 = (3,2,4,0,1)(\bar{3},\bar{2},\bar{4},\bar{0},\bar{1})(1,4,0)(\bar{1},\bar{4},\bar{0})2\bar{3}\bar{1}\bar{3}4$
$= (3,2,0,4,1)(\bar{3},\bar{2},\bar{0},\bar{4},\bar{1})23\underline{3\bar{1}}\bar{3}4 = (3,2,0,4,1)(\bar{3},\bar{2},\bar{0},\bar{4},\bar{1})23(2,0,4)(\bar{2},\bar{0},\bar{4})1\bar{3}\bar{3}4$
$= (3,0,2,4,1)(\bar{3},\bar{0},\bar{2},\bar{4},\bar{1})03134 \in [01214]$. □

Lemma-22 ($0121\bar{0} \in [01203]$)

Proof. $0121\bar{0} = 01\underline{2\bar{1}}\bar{1}\bar{0} = 01(0,3,4)(\bar{0},\bar{3},\bar{4})1\bar{2}\bar{1}\bar{0} = (0,3,4)(\bar{0},\bar{3},\bar{4})311\bar{2}\bar{1}\bar{0}$
$= (0,3,4)(\bar{0},\bar{3},\bar{4})\underline{3\bar{1}}\bar{2}\bar{1}\bar{0} = (0,3,4)(\bar{0},\bar{3},\bar{4})(2,0,4)(\bar{2},\bar{0},\bar{4})1\bar{3}\bar{2}\bar{1}\bar{0}$
$= (0,3,2)(\bar{0},\bar{3},\bar{2})1\underline{\bar{3}\bar{2}\bar{1}\bar{0}} = (0,3,2)(\bar{0},\bar{3},\bar{2})1(3,0,2,1,4)(\bar{3},\bar{0},\bar{2},\bar{1},\bar{4})1234$
$= (3,4,1)(\bar{3},\bar{4},\bar{1})31234 \in [01203]$. □

Lemma-23 ($01230 \in [01\bar{2}\bar{0}]$)

Proof. $\underline{0123}0 = (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{2}\bar{1}\bar{0}\underline{\bar{4}0}$
$= (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{2}\bar{1}\bar{0}(2,3,1)(\bar{2},\bar{3},\bar{1})\bar{0}4 = (2,0,1,4,3)(\bar{2},\bar{0},\bar{1},\bar{4},\bar{3})\bar{3}\bar{2}\bar{0}\bar{0}4$
$= (2,0,1,4,3)(\bar{2},\bar{0},\bar{1},\bar{4},\bar{3})\bar{3}\underline{\bar{2}0}4 = (2,0,1,4,3)(\bar{2},\bar{0},\bar{1},\bar{4},\bar{3})\bar{3}$
$(1,3,4)(\bar{1},\bar{3},\bar{4})\bar{0}24 = (2,0,3)(\bar{2},\bar{0},\bar{3})\bar{4}\bar{0}24 \in [01\bar{2}\bar{0}]$. □

Lemma-24 ($01231 \in [01204]$)

Proof. $\underline{0123}1 = (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{2}\bar{1}\bar{0}\underline{\bar{4}1}$
$= (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{2}\bar{1}\bar{0}(0,3,2)(\bar{0},\bar{3},\bar{2})\bar{1}4 = (2,3,1,4,0)(\bar{2},\bar{3},\bar{1},\bar{4},\bar{0})\bar{0}\bar{1}\bar{3}\bar{1}4$
$= (2,3,1,4,0)(\bar{2},\bar{3},\bar{1},\bar{4},\bar{0})\bar{0}1\underline{1\bar{3}}\bar{1}4 = (2,3,1,4,0)(\bar{2},\bar{3},\bar{1},\bar{4},\bar{0})\bar{0}1(2,4,0)(\bar{2},\bar{4},\bar{0})3\bar{1}\bar{1}4$
$= (2,3,1,4,0)(\bar{2},\bar{3},\bar{1},\bar{4},\bar{0})\underline{\bar{2}1}314 = (2,3,1,4,0)(\bar{2},\bar{3},\bar{1},\bar{4},\bar{0})(3,0,4)(\bar{3},\bar{0},\bar{4})\bar{1}2314$
$= (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{1}\underline{2314} = (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{1}(1,2,4,3,0)(\bar{1},\bar{2},\bar{4},\bar{3},\bar{0})\bar{1}\bar{3}\bar{2}\bar{0}$
$= (2,1,3)(\bar{2},\bar{1},\bar{3})\bar{2}\bar{1}\bar{3}\bar{2}\bar{0} \in [01204]$. □

Lemma-25 ($01232 \in [0120\bar{3}]$)

Proof. $01232 = 01\underline{2\bar{3}}\bar{3}2 = 01(1,0,4)(\bar{1},\bar{0},\bar{4})3\bar{2}\underline{\bar{3}2} = 01(1,0,4)(\bar{1},\bar{0},\bar{4})3\bar{2}(1,0,4)(\bar{1},\bar{0},\bar{4})\bar{2}3$
$= (1,4,0)(\bar{1},\bar{4},\bar{0})143\bar{2}\bar{2}3 = (1,4,0)(\bar{1},\bar{4},\bar{0})\underline{1432}3$
$= (1,4,0)(\bar{1},\bar{4},\bar{0})(1,2,4,0,3)(\bar{1},\bar{2},\bar{4},\bar{0},\bar{3})\bar{3}\bar{4}\bar{1}\bar{0}3 = (1,0,2,4,3)(\bar{1},\bar{0},\bar{2},\bar{4},\bar{3})\bar{3}\bar{4}\bar{1}\underline{\bar{0}3}$
$= (1,0,2,4,3)(\bar{1},\bar{0},\bar{2},\bar{4},\bar{3})\bar{3}\bar{4}\bar{1}(1,2,4)(\bar{1},\bar{2},\bar{4})\bar{3}0$
$= (1,0,4,3,2)(\bar{1},\bar{0},\bar{4},\bar{3},\bar{2})\bar{3}\bar{1}\bar{2}\bar{3}0 \in [0120\bar{3}]$. □

Lemma-26 ($01234 \in [012]$)

Proof. $\underline{0123}4 = (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{2}\bar{1}\bar{0}\bar{4}4 = (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{2}\bar{1}\bar{0} \in [012]$. □

Lemma-27 ($0123\bar{0} \in [0120\bar{3}]$)

Proof. $012\underline{3\bar{0}} = 012(1,2,4)(\bar{1},\bar{2},\bar{4})0\bar{3} = (1,2,4)(\bar{1},\bar{2},\bar{4})0240\bar{3} \in [0120\bar{3}]$. □

Lemma-28 ($0123\bar{1} \in [01203]$)

Proof. $012\underline{3\bar{1}} = 012(2,0,4)(\bar{2},\bar{0},\bar{4})1\bar{3} = (2,0,4)(\bar{2},\bar{0},\bar{4})4101\bar{3} = (2,0,4)(\bar{2},\bar{0},\bar{4})41\underline{0\bar{1}}\bar{1}\bar{3}$
$= (2,0,4)(\bar{2},\bar{0},\bar{4})41(2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}\bar{1}\bar{3} = (2,0,3)(\bar{2},\bar{0},\bar{3})311\bar{0}\bar{1}\bar{3}$
$= (2,0,3)(\bar{2},\bar{0},\bar{3})3\bar{1}\bar{0}\bar{1}\bar{3} = (2,0,3)(\bar{2},\bar{0},\bar{3})3\underline{\bar{1}0}0\bar{1}\bar{3}$
$= (2,0,3)(\bar{2},\bar{0},\bar{3})3(2,4,3)(\bar{2},\bar{4},\bar{3})\bar{0}10\bar{1}\bar{3} = (2,0)(3,4)(\bar{2},\bar{0})(\bar{3},\bar{4})2\bar{0}1\underline{0\bar{1}}\bar{3}$
$= (2,0)(3,4)(\bar{2},\bar{0})(\bar{3},\bar{4})2\bar{0}1(2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}\bar{3} = (2,0,4)(\bar{2},\bar{0},\bar{4})4\bar{0}11\bar{0}\bar{3}$
$= (2,0,4)(\bar{2},\bar{0},\bar{4})\underline{4\bar{0}}\bar{1}\bar{0}\bar{3} = (2,0,4)(\bar{2},\bar{0},\bar{4})(1,3,2)(\bar{1},\bar{3},\bar{2})0\bar{4}\bar{1}\bar{0}\bar{3}$
$= (2,0,4,1,3)(\bar{2},\bar{0},\bar{4},\bar{1},\bar{3})0\underline{\bar{4}\bar{1}\bar{0}\bar{3}}$
$= (2,0,4,1,3)(\bar{2},\bar{0},\bar{4},\bar{1},\bar{3})0(4,3,1,2,0)(\bar{4},\bar{3},\bar{1},\bar{2},\bar{0})0142$
$= (2,4)(0,3)(\bar{2},\bar{4})(\bar{0},\bar{3})40142 \in [01203].$ □

Lemma-29 ($0123\bar{2} \in [01\bar{2}\bar{3}]$)

Proof. $012\underline{3\bar{2}} = 012(1,4,0)(\bar{1},\bar{4},\bar{0})2\bar{3} = (1,4,0)(\bar{1},\bar{4},\bar{0})1422\bar{3}$
$= (1,4,0)(\bar{1},\bar{4},\bar{0})14\bar{2}\bar{3} \in [01\bar{2}\bar{3}].$ □

Lemma-30 ($0123\bar{4} \in [012\bar{3}]$)

Proof. $\underline{0123}\bar{4} = (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{2}\bar{1}\bar{0}4\bar{4} = (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{2}\bar{1}\underline{\bar{0}4}$
$= (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{2}\bar{1}(2,1,3)(\bar{2},\bar{1},\bar{3})\bar{4}0 = (2,0)(1,4)(\bar{2},\bar{0})(\bar{1},\bar{4})\bar{1}\bar{3}\bar{4}0 \in [012\bar{3}].$ □

Lemma-31 ($012\bar{3}0 \in [0\bar{1}\bar{2}3]$)

Proof. $012\bar{3}0 = \underline{0123}30 = (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{2}\bar{1}\bar{0}\underline{\bar{4}3}0$
$= (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{2}\bar{1}\bar{0}(2,1,0)(\bar{2},\bar{1},\bar{0})\bar{3}40 = (3,0)(1,4)(\bar{3},\bar{0})(\bar{1},\bar{4})\underline{\bar{1}0\bar{2}\bar{3}}40$
$= (3,0)(1,4)(\bar{3},\bar{0})(\bar{1},\bar{4})(2,1,3,0,4)(\bar{2},\bar{1},\bar{3},\bar{0},\bar{4})201440 = (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})201\bar{4}0$
$= (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})2\bar{0}\underline{01}\bar{4}0 = (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})2\bar{0}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}0\bar{4}0$
$= (1,3,2)(\bar{1},\bar{3},\bar{2})3\bar{0}\bar{1}\underline{0\bar{4}}0 = (1,3,2)(\bar{1},\bar{3},\bar{2})3\bar{0}\bar{1}(1,2,3)(\bar{1},\bar{2},\bar{3})4\bar{0}0$
$= \underline{1\bar{0}}\bar{2}4 = (2,3,4)(\bar{2},\bar{3},\bar{4})0\bar{1}\underline{\bar{2}4} = (2,3,4)(\bar{2},\bar{3},\bar{4})0\bar{1}(0,3,1)(\bar{0},\bar{3},\bar{1})\bar{4}2$
$= (2,1,0,3,4)(\bar{2},\bar{1},\bar{0},\bar{3},\bar{4})3\bar{0}\bar{4}2 \in [0\bar{1}\bar{2}3].$ □

Lemma-32 ($012\bar{3}4 \in [012]$)

Proof. $012\bar{3}4 = \underline{0123}34 = (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{2}\bar{1}\bar{0}\underline{\bar{4}3}4$
$= (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{2}\bar{1}\bar{0}(2,1,0)(\bar{2},\bar{1},\bar{0})\bar{3}44 = (0,3)(1,4)(\bar{0},\bar{3})(\bar{1},\bar{4})\bar{1}\underline{0\bar{2}\bar{3}4}$
$= (0,3)(1,4)(\bar{0},\bar{3})(\bar{1},\bar{4})\bar{1}(3,0,4,2,1)(\bar{3},\bar{0},\bar{4},\bar{2},\bar{1})3201$
$= (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})\bar{3}3201 = (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})201 \in [012].$ □

Lemma-33 $(012\bar3\bar4 \in [0123])$

Proof. $012\bar3\bar4 = \underline{0123}3\bar4 = (2,0,3,1,4)(\bar2,\bar0,\bar3,\bar1,\bar4)\bar2\bar1\bar0\underline{\bar43}\bar4$
$= (2,0,3,1,4)(\bar2,\bar0,\bar3,\bar1,\bar4)\bar2\bar1\bar0(2,1,0)(\bar2,\bar1,\bar0)\bar34\bar4$
$= (0,3)(4,1)(\bar0,\bar3)(\bar4,\bar1)\bar1\bar0\bar2\bar3 \in [0123].$ □

Lemma-34 $(01\bar2\bar01 \in [0123])$

Proof. $0\underline{1\bar2}\bar01 = 0(0,4,3)(\bar0,\bar4,\bar3)2\bar1\bar01 = (0,4,3)(\bar0,\bar4,\bar3)42\bar1\underline{\bar01}$
$= (0,4,3)(\bar0,\bar4,\bar3)42\bar1(2,3,4)(\bar2,\bar3,\bar4)\bar10 = (0,2,3)(\bar0,\bar2,\bar3)23\bar1\bar10$
$= (0,2,3)(\bar0,\bar2,\bar3)\underline{2310} = (0,2,3)(\bar0,\bar2,\bar3)(2,0,3,4,1)(\bar2,\bar0,\bar3,\bar4,\bar1)\bar1\bar3\bar2\bar4$
$= (2,4,1)(\bar2,\bar4,\bar1)\bar1\bar3\bar2\bar4 \in [0123].$ □

Lemma-35 $(01\bar2\bar02 \in [0120])$

Proof. $01\bar2\underline{\bar02} = 01\bar2(1,4,3)(\bar1,\bar4,\bar3)\bar20 = (1,4,3)(\bar1,\bar4,\bar3)04\bar2\bar20$
$= (1,4,3)(\bar1,\bar4,\bar3)0420 \in [0120].$ □

Lemma-36 $(01\bar2\bar03 \in [0120\bar3])$

Proof. $01\bar2\bar03 = \bar0\underline{01}\bar2\bar03 = \bar0(2,3,4)(\bar2,\bar3,\bar4)\bar10\bar2\bar03 = (2,3,4)(\bar2,\bar3,\bar4)\bar0\bar1\underline{0\bar2}\bar03$
$= (2,3,4)(\bar2,\bar3,\bar4)\bar0\bar1(1,3,4)(\bar1,\bar3,\bar4)2\bar0\bar03 = (2,4)(3,1)(\bar2,\bar4)(\bar3,\bar1)\bar0\underline{\bar32}03$
$= (2,4)(3,1)(\bar2,\bar4)(\bar3,\bar1)\bar0(0,4,1)(\bar0,\bar4,\bar1)\bar2303 = (2,1,3,0,4)(\bar2,\bar1,\bar3,\bar0,\bar4)\bar4\bar2303$
$= (2,1,3,0,4)(\bar2,\bar1,\bar3,\bar0,\bar4)\bar4\bar2\bar3\underline{30}3 = (2,1,3,0,4)(\bar2,\bar1,\bar3,\bar0,\bar4)\bar4\bar2\bar3(1,4,2)(\bar1,\bar4,\bar2)\bar033$
$= (2,4,1,3,0)(\bar2,\bar4,\bar1,\bar3,\bar0)\underline{\bar2\bar1\bar3\bar0}\bar3$
$= (2,4,1,3,0)(\bar2,\bar4,\bar1,\bar3,\bar0)(2,0,1,4,3)(\bar2,\bar0,\bar1,\bar4,\bar3)3124\bar3$
$= (2,3,1)(\bar2,\bar3,\bar1)312\underline{4\bar3} = (2,3,1)(\bar2,\bar3,\bar1)312(2,0,1)(\bar2,\bar0,\bar1)3\bar4$
$= (2,3,1,0,4)(\bar2,\bar3,\bar1,\bar0,\bar4)3203\bar4 \in [0120\bar3].$ □

Lemma-37 $(01\bar2\bar04 \in [012\bar3\bar0])$

Proof. $01\bar2\bar04 = 01\underline{\bar20}04 = 01(1,3,4)(\bar1,\bar3,\bar4)\bar0204 = (1,3,4)(\bar1,\bar3,\bar4)0\underline{3\bar0}204$
$= (1,3,4)(\bar1,\bar3,\bar4)0(2,4,1)(\bar2,\bar4,\bar1)0\bar3204 = (2,4)(3,1)(\bar2,\bar4)(\bar3,\bar1)00\bar3204$
$= (2,4)(3,1)(\bar2,\bar4)(\bar3,\bar1)\bar0\underline{\bar32}04 = (2,4)(3,1)(\bar2,\bar4)(\bar3,\bar1)\bar0(1,0,4)(\bar1,\bar0,\bar4)\bar2304$
$= (4,2,1,3,0)(\bar4,\bar2,\bar1,\bar3,\bar0)\bar4\bar2304 = (4,2,1,3,0)(\bar4,\bar2,\bar1,\bar3,\bar0)\bar42\underline{2304}$
$= (4,2,1,3,0)(\bar4,\bar2,\bar1,\bar3,\bar0)\bar42(2,4,3,1,0)(\bar2,\bar4,\bar3,\bar1,\bar0)\bar0\bar3\bar2\bar1$
$= (2,0,3)(\bar2,\bar0,\bar3)\underline{\bar34}\bar0\bar3\bar2\bar1 = (2,0,3)(\bar2,\bar0,\bar3)(2,0,1)(\bar2,\bar0,\bar1)\bar43\bar0\bar3\bar2\bar1$

$= (2,1)(3,0)(\bar2,\bar1)(\bar3,\bar0)\bar4\underline{3\bar0}\bar3\bar2\bar1 = (2,1)(3,0)(\bar2,\bar1)(\bar3,\bar0)\bar4(1,2,4)(\bar1,\bar2,\bar4)0\bar3\bar3\bar2\bar1$

$= (4,1)(3,0)(\bar4,\bar1)(\bar3,\bar0)\bar1 03\bar2\bar1 = (4,1)(3,0)(\bar4,\bar1)(\bar3,\bar0)1\underline{1032}2\bar1$

$= (4,1)(3,0)(\bar4,\bar1)(\bar3,\bar0)1(3,1,2,0,4)(\bar3,\bar1,\bar2,\bar0,\bar4)\bar3\bar0\bar1\bar4 2\bar1$

$= (1,3,4,2,0)(\bar1,\bar3,\bar4,\bar2,\bar0)2\bar3\bar0\bar1\underline{\bar4 2}\bar1 = (1,3,4,2,0)(\bar1,\bar3,\bar4,\bar2,\bar0)2\bar3\bar0\bar1(1,3,0)(\bar1,\bar3,\bar0)\bar2 4\bar1$

$= (1,0,3,4,2)(\bar1,\bar0,\bar3,\bar4,\bar2)2\underline{\bar0\bar1\bar3\bar2}4\bar1$

$= (1,0,3,4,2)(\bar1,\bar0,\bar3,\bar4,\bar2)2(1,4,3,0,2)(\bar1,\bar4,\bar3,\bar0,\bar2)31044\bar1 = (1,2,4)(\bar1,\bar2,\bar4)1310\bar4\bar1$

$= (1,2,4)(\bar1,\bar2,\bar4)\bar1\underline{\bar1 3}10\bar4\bar1 = (1,2,4)(\bar1,\bar2,\bar4)\bar1(2,0,4)(\bar2,\bar0,\bar4)\bar3 110\bar4\bar1$

$= (1,0,4)(\bar1,\bar0,\bar4)\bar1\bar3\underline{\bar1 0}\bar4\bar1 = (1,0,4)(\bar1,\bar0,\bar4)\bar1\bar3(3,2,4)(\bar3,\bar2,\bar4)\bar0 1\bar4\bar1$

$= (1,0,3,2,4)(\bar1,\bar0,\bar3,\bar2,\bar4)\bar1\bar2\bar0\underline{1\bar4}\bar1 = (1,0,3,2,4)(\bar1,\bar0,\bar3,\bar2,\bar4)\bar1\bar2\bar0(2,0,3)(\bar2,\bar0,\bar3)4\bar1\bar1$

$= (1,3,0,2,4)(\bar1,\bar3,\bar0,\bar2,\bar4)\bar1 0\bar3 41 \in [012\bar3\bar0].$ $\square$

Lemma-38 $(01\bar2\bar3 1 \in [0123])$

Proof. $0\underline{1\bar2}\bar3 1 = 0(0,4,3)(\bar0,\bar4,\bar3)2\bar1\bar3 1 = (0,4,3)(\bar0,\bar4,\bar3)42\bar1\underline{\bar3 1}$

$= (0,4,3)(\bar0,\bar4,\bar3)42\bar1(2,4,0)(\bar2,\bar4,\bar0)\bar1 3 = (4,3,2)(\bar4,\bar3,\bar2)04\bar1\bar1 3$

$= (4,3,2)(\bar4,\bar3,\bar2)0413 \in [0123].$ $\square$

Lemma-39 $(01\bar2\bar3\bar1 \in [0120\bar3])$

Proof. $0\underline{1\bar2}\bar3\bar1 = 0(0,4,3)(\bar0,\bar4,\bar3)2\bar1\bar3\bar1 = (0,4,3)(\bar0,\bar4,\bar3)42\bar1\bar3\bar1 = (0,4,3)(\bar0,\bar4,\bar3)421\underline{1\bar3}\bar1$

$= (0,4,3)(\bar0,\bar4,\bar3)421(2,4,0)(\bar2,\bar4,\bar0)3\bar1\bar1 = (4,3,2)(\bar4,\bar3,\bar2)\underline{0413}1$

$= (4,3,2)(\bar4,\bar3,\bar2)(1,0,3,4,2)(\bar1,\bar0,\bar3,\bar4,\bar2)\bar1\bar4\bar0 21 = (3,1,0)(\bar3,\bar1,\bar0)\bar1\bar4\bar0\underline{21}$

$= (3,1,0)(\bar3,\bar1,\bar0)\bar1\bar4\bar0(3,0,4)(\bar3,\bar0,\bar4)\bar1 2 = (3,1,4)(\bar3,\bar1,\bar4)\bar1\bar3\bar4\bar1 2$

$= (3,1,4)(\bar3,\bar1,\bar4)\bar1\underline{\bar3\bar4\bar1\bar2}\bar2 = (3,1,4)(\bar3,\bar1,\bar4)\bar1(3,2,4,0,1)(\bar3,\bar2,\bar4,\bar0,\bar1)1430\bar2$

$= (1,0)(4,2)(\bar1,\bar0)(\bar4,\bar2)\bar3 143\underline{0\bar2} = (1,0)(4,2)(\bar1,\bar0)(\bar4,\bar2)\bar3 143(1,3,4)(\bar1,\bar3,\bar4)2\bar0$

$= (1,0,3,4,2)(\bar1,\bar0,\bar3,\bar4,\bar2)\bar4\underline{3142}\bar0$

$= (1,0,3,4,2)(\bar1,\bar0,\bar3,\bar4,\bar2)\bar4(3,2,1,0,4)(\bar3,\bar2,\bar1,\bar0,\bar4)\bar4\bar1\bar3\bar0\bar0$

$= (1,4)(0,2)(\bar1,\bar4)(\bar0,\bar2)\bar3\bar4\bar1\bar3 0 \in [0120\bar3].$ $\square$

Lemma-40 $(0\bar1\bar2 31 \in [012\bar3\bar0])$

Proof. $0\bar1\bar2 31 = 01\underline{1\bar2}3\bar1 = 01(0,4,3)(\bar0,\bar4,\bar3)2\bar1 31 = (0,4,3)(\bar0,\bar4,\bar3)412\underline{\bar1 3}1$

$= (0,4,3)(\bar0,\bar4,\bar3)412(2,0,4)(\bar2,\bar0,\bar4)\bar3 11 = (0,2)(4,3)(\bar0,\bar2)(\bar4,\bar3)21\underline{0\bar3}\bar1$

$= (0,2)(4,3)(\bar0,\bar2)(\bar4,\bar3)21(1,4,2)(\bar1,\bar4,\bar2)3\bar0\bar1$

$= (0,1,4,3,2)(\bar0,\bar1,\bar4,\bar3,\bar2)143\bar0\bar1 \in [012\bar3\bar0].$ $\square$

Lemma-41 $(0\bar{1}\bar{2}3\bar{1} \in [012\bar{3}])$

Proof. $0\bar{1}\bar{2}3\bar{1} = 01\underline{1\bar{2}}3\bar{1} = 01(0,4,3)(\bar{0},\bar{4},\bar{3})2\bar{1}3\bar{1} = (0,4,3)(\bar{0},\bar{4},\bar{3})412\underline{\bar{1}3}\bar{1}$
$= (0,4,3)(\bar{0},\bar{4},\bar{3})412(2,0,4)(\bar{2},\bar{0},\bar{4})\bar{3}1\bar{1} = (0,2)(4,3)(\bar{0},\bar{2})(\bar{4},\bar{3})21\underline{0\bar{3}}$
$= (0,2)(4,3)(\bar{0},\bar{2})(\bar{4},\bar{3})21(1,4,2)(\bar{1},\bar{4},\bar{2})3\bar{0} = (0,1,4,3,2)(\bar{0},\bar{1},\bar{4},\bar{3},\bar{2})143\bar{0} \in [012\bar{3}]$. □

Lemma-42 $(0\bar{1}\bar{2}30 \in [01023])$

Proof. $\underline{0\bar{1}}\bar{2}30 = (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}\bar{2}30 = (2,4,3)(\bar{2},\bar{4},\bar{3})10\underline{0\bar{2}}30$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})10(1,3,4)(\bar{1},\bar{3},\bar{4})2\bar{0}30 = (2,1,3)(\bar{2},\bar{1},\bar{3})302\underline{\bar{0}3}0$
$= (2,1,3)(\bar{2},\bar{1},\bar{3})302(1,2,4)(\bar{1},\bar{2},\bar{4})\bar{3}00 = (1,3,4)(\bar{1},\bar{3},\bar{4})30\underline{4\bar{3}}\bar{0}$
$= (1,3,4)(\bar{1},\bar{3},\bar{4})30(2,0,1)(\bar{2},\bar{0},\bar{1})3\bar{4}\bar{0} = (1,3,4,2,0)(\bar{1},\bar{3},\bar{4},\bar{2},\bar{0})313\bar{4}\bar{0}$
$= (1,3,4,2,0)(\bar{1},\bar{3},\bar{4},\bar{2},\bar{0})\bar{3}\underline{\bar{3}1}3\bar{4}\bar{0} = (1,3,4,2,0)(\bar{1},\bar{3},\bar{4},\bar{2},\bar{0})\bar{3}(2,4,0)(\bar{2},\bar{4},\bar{0})\bar{1}33\bar{4}\bar{0}$
$= (1,3,0)(\bar{1},\bar{3},\bar{0})\bar{3}\bar{1}\bar{3}\bar{4}\bar{0} = (1,3,0)(\bar{1},\bar{3},\bar{0})\bar{3}1\underline{1\bar{3}}4\bar{0}$
$= (1,3,0)(\bar{1},\bar{3},\bar{0})\bar{3}1(2,4,0)(\bar{2},\bar{4},\bar{0})3\bar{1}\bar{4}\bar{0} = (1,3,2,4,0)(\bar{1},\bar{3},\bar{2},\bar{4},\bar{0})\underline{\bar{3}1}3\bar{1}\bar{4}\bar{0}$
$= (1,3,2,4,0)(\bar{1},\bar{3},\bar{2},\bar{4},\bar{0})(2,4,0)(\bar{2},\bar{4},\bar{0})\bar{1}33\bar{1}\bar{4}\bar{0}$
$= (1,3,4,2,0)(\bar{1},\bar{3},\bar{4},\bar{2},\bar{0})\bar{1}3\bar{1}\bar{4}\bar{0} \in [01023]$. □

Lemma-43 $(0\bar{1}\bar{2}3\bar{0} \in [0102])$

Proof. $\underline{0\bar{1}}\bar{2}3\bar{0} = (2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}\bar{2}3\bar{0} = (2,4,3)(\bar{2},\bar{4},\bar{3})10\underline{0\bar{2}}3\bar{0}$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})10(1,3,4)(\bar{1},\bar{3},\bar{4})2\bar{0}3\bar{0} = (2,1,3)(\bar{2},\bar{1},\bar{3})302\underline{\bar{0}3}\bar{0}$
$= (2,1,3)(\bar{2},\bar{1},\bar{3})302(1,2,4)(\bar{1},\bar{2},\bar{4})\bar{3}0\bar{0} = (1,3,4)(\bar{1},\bar{3},\bar{4})304\bar{3}$
$= (1,3,4)(\bar{1},\bar{3},\bar{4})\bar{3}\underline{\bar{3}0}4\bar{3} = (1,3,4)(\bar{1},\bar{3},\bar{4})\bar{3}(1,4,2)(\bar{1},\bar{4},\bar{2})\bar{0}34\bar{3}$
$= (1,3,2)(\bar{1},\bar{3},\bar{2})\bar{3}034\bar{3} = (1,3,2)(\bar{1},\bar{3},\bar{2})\bar{3}\bar{0}\bar{3}\underline{34}\bar{3}$
$= (1,3,2)(\bar{1},\bar{3},\bar{2})\bar{3}\bar{0}\bar{3}(2,0,1)(\bar{2},\bar{0},\bar{1})\bar{4}3\bar{3} = (1,3,0)(\bar{1},\bar{3},\bar{0})\bar{3}\bar{1}3\bar{4} \in [0102]$. □

Lemma-44 $(0\bar{1}\bar{2}\bar{3}\bar{4} \in [0\bar{1}\bar{2}\bar{3}])$

Proof. $0\underline{\bar{1}\bar{2}\bar{3}\bar{4}} = 0(1,4,2,0,3)(\bar{1},\bar{4},\bar{2},\bar{0},\bar{3})3210 = (1,4,2,0,3)(\bar{1},\bar{4},\bar{2},\bar{0},\bar{3})33210$
$= (1,4,2,0,3)(\bar{1},\bar{4},\bar{2},\bar{0},\bar{3})\underline{\bar{3}2}10 = (1,4,2,0,3)(\bar{1},\bar{4},\bar{2},\bar{0},\bar{3})(0,4,1)(\bar{0},\bar{4},\bar{1})\bar{2}310$
$= (4,2)(3,0)(\bar{4},\bar{2})(\bar{3},\bar{0})\bar{2}310 \in [0\bar{1}\bar{2}\bar{3}]$. □

Lemma-45 $(0\bar{1}\bar{2}\bar{3}0 \in [01214])$

Proof. $0\bar{1}\bar{2}\bar{3}0 = \bar{0}\underline{0\bar{1}\bar{2}\bar{3}}0 = \bar{0}(0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})21040$
$= (0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})\underline{\bar{3}2}1040 = (0,3,1,4,2)(\bar{0},\bar{3},\bar{1},\bar{4},\bar{2})(0,4,1)(\bar{0},\bar{4},\bar{1})\bar{2}31040$

$= (0,3)(4,2)(\bar0,\bar3)(\bar4,\bar2)\bar2\underline{3104}0 = (0,3)(4,2)(\bar0,\bar3)(\bar4,\bar2)\bar2(3,4,1,2,0)(\bar3,\bar4,\bar1,\bar2,\bar0)\bar0\bar1\bar3\bar20$

$= (0,4)(1,2)(\bar0,\bar4)(\bar1,\bar2)\bar0\bar0\bar1\bar3\bar20 = (0,4)(1,2)(\bar0,\bar4)(\bar1,\bar2)\underline{0\bar1}\bar3\bar20$

$= (0,4)(1,2)(\bar0,\bar4)(\bar1,\bar2)(2,4,3)(\bar2,\bar4,\bar3)1\bar0\bar3\bar20 = (0,3,2,1,4)(\bar0,\bar3,\bar2,\bar1,\bar4)10\underline{0\bar3}\bar20$

$= (0,3,2,1,4)(\bar0,\bar3,\bar2,\bar1,\bar4)10(1,4,2)(\bar1,\bar4,\bar2)3\bar0\bar20 = (0,3,1,2,4)(\bar0,\bar3,\bar1,\bar2,\bar4)403\bar0\underline{\bar20}$

$= (0,3,1,2,4)(\bar0,\bar3,\bar1,\bar2,\bar4)403\bar0(1,3,4)(\bar1,\bar3,\bar4)\bar02$

$= (0,4)(1,2)(\bar0,\bar4)(\bar1,\bar2)10402 \in [01214].$ □

Lemma-46 $(0\bar1\bar2\bar3\bar0 \in [01023])$

Proof. $0\underline{\bar1\bar2\bar3\bar0} = 0(3,1,0,2,4)(\bar3,\bar1,\bar0,\bar2,\bar4)3214$

$= (3,1,0,2,4)(\bar3,\bar1,\bar0,\bar2,\bar4)23214 \in [01023].$ □

Lemma-47 $(010230 \in [012\bar3\bar0])$

Proof. $0\underline{1023}0 = 0(1,3,0,4,2)(\bar1,\bar3,\bar0,\bar4,\bar2)\bar2\bar0\bar1\bar40 = (1,3,0,4,2)(\bar1,\bar3,\bar0,\bar4,\bar2)4\bar2\bar0\bar1\underline{\bar40}$

$= (1,3,0,4,2)(\bar1,\bar3,\bar0,\bar4,\bar2)4\bar2\bar0\bar1(3,1,2)(\bar3,\bar1,\bar2)\bar04 = (0,4,3)(\bar0,\bar4,\bar3)4\bar3\bar0\bar2\bar04$

$= (0,4,3)(\bar0,\bar4,\bar3)\bar4\underline{4\bar3\bar0\bar2}\bar04 = (0,4,3)(\bar0,\bar4,\bar3)\bar4(3,1,0,4,2)(\bar3,\bar1,\bar0,\bar4,\bar2)0341\bar04$

$= (0,2,3,4,1)(\bar0,\bar2,\bar3,\bar4,\bar1)\underline{\bar20}341\bar04 = (0,2,3,4,1)(\bar0,\bar2,\bar3,\bar4,\bar1)(1,3,4)(\bar1,\bar3,\bar4)\bar02341\bar04$

$= (0,2,4,3,1)(\bar0,\bar2,\bar4,\bar3,\bar1)\bar0\underline{2341}\bar04$

$= (0,2,4,3,1)(\bar0,\bar2,\bar4,\bar3,\bar1)\bar0(2,1,3,0,4)(\bar2,\bar1,\bar3,\bar0,\bar4)\bar4\bar3\bar2\bar0\bar04$

$= (0,1,4)(\bar0,\bar1,\bar4)\bar4\bar4\bar3\bar2\bar0\bar04 = (0,1,4)(\bar0,\bar1,\bar4)4\bar3\underline{\bar20}4$

$= (0,1,4)(\bar0,\bar1,\bar4)4\bar3(1,3,4)(\bar1,\bar3,\bar4)\bar024 = (0,3,4)(\bar0,\bar3,\bar4)1\bar4\bar024$

$= (0,3,4)(\bar0,\bar3,\bar4)14\underline{4\bar0}24 = (0,3,4)(\bar0,\bar3,\bar4)14(1,3,2)(\bar1,\bar3,\bar2)0\bar424$

$= (0,2,1,3,4)(\bar0,\bar2,\bar1,\bar3,\bar4)340\underline{\bar42}4 = (0,2,1,3,4)(\bar0,\bar2,\bar1,\bar3,\bar4)340(0,1,3)(\bar0,\bar1,\bar3)\bar244$

$= (0,2,3,4,1)(\bar0,\bar2,\bar3,\bar4,\bar1)04\underline{1\bar2}\bar4 = (0,2,3,4,1)(\bar0,\bar2,\bar3,\bar4,\bar1)04(0,4,3)(\bar0,\bar4,\bar3)2\bar1\bar4$

$= (0,2)(4,1)(\bar0,\bar2)(\bar4,\bar1)432\bar1\bar4 \in [012\bar3\bar0].$ □

Lemma-48 $(010231 \in [01023])$

Proof. $01\underline{0231} = 01(3,0,1,2,4)(\bar3,\bar0,\bar1,\bar2,\bar4)\bar3\bar2\bar0\bar4 = (3,0,1,2,4)(\bar3,\bar0,\bar1,\bar2,\bar4)1\underline{2\bar3}\bar2\bar0\bar4$

$= (3,0,1,2,4)(\bar3,\bar0,\bar1,\bar2,\bar4)1(1,0,4)(\bar1,\bar0,\bar4)3\bar2\bar2\bar0\bar4 = (3,4)(1,2)(\bar3,\bar4)(\bar1,\bar2)032\bar0\bar4$

$= (3,4)(1,2)(\bar3,\bar4)(\bar1,\bar2)\bar0\underline{\bar03}2\bar0\bar4 = (3,4)(1,2)(\bar3,\bar4)(\bar1,\bar2)\bar0(1,2,4)(\bar1,\bar2,\bar4)\bar302\bar0\bar4$

$= (3,1,4)(\bar3,\bar1,\bar4)\bar0\bar302\bar0\bar4 = (3,1,4)(\bar3,\bar1,\bar4)\bar0\bar30\underline{\bar02}\bar0\bar4$

$= (3,1,4)(\bar3,\bar1,\bar4)\bar0\bar3\bar0(3,1,4)(\bar3,\bar1,\bar4)\bar2\bar0\bar0\bar4 = (3,4,1)(\bar3,\bar4,\bar1)\bar0\bar1\bar02\bar4 \in [01023].$ □

Lemma-49 $(0102\bar3 \in [0\bar1\bar23])$

Proof. $0102\bar{3} = 0\underline{1023}3 = 0(1,3,0,4,2)(\bar1,\bar3,\bar0,\bar4,\bar2)\bar2\bar0\bar1\bar4 3$
$= (1,3,0,4,2)(\bar1,\bar3,\bar0,\bar4,\bar2)4\bar2\bar0\bar1\underline{\bar4 3} = (1,3,0,4,2)(\bar1,\bar3,\bar0,\bar4,\bar2)4\bar2\bar0\bar1(2,1,0)(\bar2,\bar1,\bar0)\bar3 4$
$= (1,3,2,0,4)(\bar1,\bar3,\bar2,\bar0,\bar4)4\underline{\bar1\bar2\bar0\bar3}4$
$= (1,3,2,0,4)(\bar1,\bar3,\bar2,\bar0,\bar4)4(1,3,2,4,0)(\bar1,\bar3,\bar2,\bar4,\bar0)02144$
$= (1,2)(3,4)(\bar1,\bar2)(\bar3,\bar4)002144 = (1,2)(3,4)(\bar1,\bar2)(\bar3,\bar4)\bar0 2\underline{1\bar4}$
$= (1,2)(3,4)(\bar1,\bar2)(\bar3,\bar4)\bar0 2(2,0,3)(\bar2,\bar0,\bar3)4\bar1 = (1,0,3,4,2)(\bar1,\bar0,\bar3,\bar4,\bar2)\bar3 04\bar1 \in [0\bar1\bar2 3]$. $\square$

Lemma-50 ($010234 \in [0\bar1\bar2\bar3]$)

Proof. $0\underline{1023}4 = 0(1,3,0,4,2)(\bar1,\bar3,\bar0,\bar4,\bar2)\bar2\bar0\bar1 44$
$= (1,3,0,4,2)(\bar1,\bar3,\bar0,\bar4,\bar2)4\bar2\bar0\bar1 \in [0\bar1\bar2\bar3]$. $\square$

Lemma-51 ($01023\bar0 \in [01203]$)

Proof. $01023\bar0 = \bar0\underline{\bar0 1}023\bar0 = \bar0(2,3,4)(\bar2,\bar3,\bar4)\bar1 0023\bar0 = (2,3,4)(\bar2,\bar3,\bar4)\bar0\bar1\underline{\bar0 2}3\bar0$
$= (2,3,4)(\bar2,\bar3,\bar4)\bar0\bar1(1,4,3)(\bar1,\bar4,\bar3)\bar2 03\bar0 = (2,1,4)(\bar2,\bar1,\bar4)\bar0\bar4\bar2 0\underline{3\bar0}$
$= (2,1,4)(\bar2,\bar1,\bar4)\bar0\bar4\bar2 0(1,2,4)(\bar1,\bar2,\bar4)0\bar3 = \bar0\bar1\bar4 00\bar3 = \bar0\bar1\bar4 0\bar3 \in [01203]$. $\square$

Lemma-52 ($01023\bar1 \in [01201]$)

Proof. $0\underline{1023}\bar1 = 0(1,3,0,4,2)(\bar1,\bar3,\bar0,\bar4,\bar2)\bar2\bar0\bar1\bar4\bar1 = (1,3,0,4,2)(\bar1,\bar3,\bar0,\bar4,\bar2)4\bar2\bar0\bar1\bar4\bar1$
$= (1,3,0,4,2)(\bar1,\bar3,\bar0,\bar4,\bar2)4\bar2\bar0 1\underline{1\bar4}\bar1 = (1,3,0,4,2)(\bar1,\bar3,\bar0,\bar4,\bar2)4\bar2\bar0 1(2,0,3)(\bar2,\bar0,\bar3)4\bar1\bar1$
$= (1,2)(0,4)(\bar1,\bar2)(\bar0,\bar4)\underline{4\bar0}\bar3 141 = (1,2)(0,4)(\bar1,\bar2)(\bar0,\bar4)(2,1,3)(\bar2,\bar1,\bar3)0\bar4\bar3 141$
$= (3,2)(0,4)(\bar1,\bar2)(\bar0,\bar4)0\bar4\bar3 141 = (3,2)(0,4)(\bar1,\bar2)(\bar0,\bar4)04\underline{4\bar3}141$
$= (3,2)(0,4)(\bar1,\bar2)(\bar0,\bar4)04(2,1,0)(\bar2,\bar1,\bar0)3\bar4 141 = (2,3,0,4,1)(\bar2,\bar3,\bar0,\bar4,\bar1)143\underline{\bar4 1}41$
$= (2,3,0,4,1)(\bar2,\bar3,\bar0,\bar4,\bar1)143(0,3,2)(\bar0,\bar3,\bar2)\bar1 441 = (0,4,1)(\bar0,\bar4,\bar1)142\bar1\underline{41}$
$= (0,4,1)(\bar0,\bar4,\bar1)142\bar1(0,3,2)(\bar0,\bar3,\bar2)\bar1 4 = (3,2,0,4,1)(\bar3,\bar2,\bar0,\bar4,\bar1)140\bar1\bar1 4$
$= (3,2,0,4,1)(\bar3,\bar2,\bar0,\bar4,\bar1)14014 \in [01201]$. $\square$

Lemma-53 ($01023\bar2 \in [01023]$)

Proof. $0102\underline{3\bar2} = 0102(0,1,4)(\bar0,\bar1,\bar4)2\bar3 = (0,1,4)(\bar0,\bar1,\bar4)14122\bar3 = (0,1,4)(\bar0,\bar1,\bar4)141\bar2\bar3$
$= (0,1,4)(\bar0,\bar1,\bar4)1\underline{4\bar1}\bar1\bar2\bar3 = (0,1,4)(\bar0,\bar1,\bar4)1(2,3,0)(\bar2,\bar3,\bar0)1\bar4\bar1\bar2\bar3$
$= (1,4,2,3,0)(\bar1,\bar4,\bar2,\bar3,\bar0)11\bar4\bar1\bar2\bar3 = (1,4,2,3,0)(\bar1,\bar4,\bar2,\bar3,\bar0)\bar1\bar4\bar1\bar2\bar3$
$= (1,4,2,3,0)(\bar1,\bar4,\bar2,\bar3,\bar0)\underline{\bar1 4}4\bar1\bar2\bar3 = (1,4,2,3,0)(\bar1,\bar4,\bar2,\bar3,\bar0)(2,3,0)(\bar2,\bar3,\bar0)\bar4 14\bar1\bar2\bar3$
$= (1,4,3,2,0)(\bar1,\bar4,\bar3,\bar2,\bar0)\bar4 14\bar1\bar2\bar3 = (1,4,3,2,0)(\bar1,\bar4,\bar3,\bar2,\bar0)\bar4 1\underline{\bar1 4}\bar1\bar2\bar3$

$= (1,4,3,2,0)(\bar{1},\bar{4},\bar{3},\bar{2},\bar{0})\bar{4}\bar{1}(2,0,3)(\bar{2},\bar{0},\bar{3})\bar{4}1\bar{1}\bar{2}\bar{3}$

$= (1,4,2,3,0)(\bar{1},\bar{4},\bar{2},\bar{3},\bar{0})\bar{4}\bar{1}\bar{4}\bar{2}\bar{3} \in [01023].$ □

Lemma-54 ($01023\bar{4} \in [01210]$)

Proof. $0102\underline{3\bar{4}} = 0102(2,1,0)(\bar{2},\bar{1},\bar{0})4\bar{3} = (2,1,0)(\bar{2},\bar{1},\bar{0})2\underline{0214}\bar{3}$

$= (2,1,0)(\bar{2},\bar{1},\bar{0})2(1,0,4,2,3)(\bar{1},\bar{0},\bar{4},\bar{2},\bar{3})\bar{1}\bar{2}\bar{0}\bar{3}\bar{3} = (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\underline{3\bar{1}}\bar{2}\bar{0}3$

$= (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})(2,0,4)(\bar{2},\bar{0},\bar{4})1\bar{3}\bar{2}\bar{0}\bar{3} = (2,4,0,3,1)(\bar{2},\bar{4},\bar{0},\bar{3},\bar{1})13\underline{3\bar{2}}\bar{0}\bar{3}$

$= (2,4,0,3,1)(\bar{2},\bar{4},\bar{0},\bar{3},\bar{1})13(1,4,0)(\bar{1},\bar{4},\bar{0})2\bar{3}\bar{0}3 = (2,0,3,4,1)(\bar{2},\bar{0},\bar{3},\bar{4},\bar{1})432\bar{3}\bar{0}3$

$= (2,0,3,4,1)(\bar{2},\bar{0},\bar{3},\bar{4},\bar{1})4323\underline{3\bar{0}}3 = (2,0,3,4,1)(\bar{2},\bar{0},\bar{3},\bar{4},\bar{1})4323(1,2,4)(\bar{1},\bar{2},\bar{4})0\bar{3}3$

$= (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})13430 \in [01214].$ □

Lemma-55 ($012012 \in [01201]$)

Proof. $012012 = \bar{0}\underline{\bar{0}1}2012 = \bar{0}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}02012 = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}02012$

$= (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}\bar{0}\underline{\bar{0}2}012 = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}\bar{0}(1,4,3)(\bar{1},\bar{4},\bar{3})\bar{2}0012$

$= (2,1,4)(\bar{2},\bar{1},\bar{4})\bar{0}\bar{4}\bar{0}\bar{2}\underline{\bar{0}1}2 = (2,1,4)(\bar{2},\bar{1},\bar{4})\bar{0}\bar{4}\bar{0}\bar{2}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}02$

$= (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})\bar{0}\underline{\bar{2}\bar{0}\bar{3}\bar{1}}02 = (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})\bar{0}(3,2,1,0,4)(\bar{3},\bar{2},\bar{1},\bar{0},\bar{4})302402$

$= (2,0,4)(\bar{2},\bar{0},\bar{4})\bar{4}302402 = (2,0,4)(\bar{2},\bar{0},\bar{4})\bar{4}3\bar{0}\underline{\bar{0}2}402$

$= (2,0,4)(\bar{2},\bar{0},\bar{4})\bar{4}3\bar{0}(3,1,4)(\bar{3},\bar{1},\bar{4})\bar{2}0402 = (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{3}1\bar{0}\bar{2}0402$

$= (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{3}1\bar{0}\bar{2}\bar{0}\underline{\bar{0}4}02 = (2,0,3,1,4)(\bar{2},\bar{0},\bar{3},\bar{1},\bar{4})\bar{3}1\bar{0}\bar{2}\bar{0}(2,1,3)(\bar{2},\bar{1},\bar{3})\bar{4}002$

$= (0,2)(1,4)(\bar{0},\bar{2})(\bar{1},\bar{4})\bar{2}\underline{3\bar{0}}\bar{1}\bar{0}\bar{4}\bar{0}2 = (0,2)(1,4)(\bar{0},\bar{2})(\bar{1},\bar{4})\bar{2}(1,2,4)(\bar{1},\bar{2},\bar{4})0\bar{3}\bar{1}\bar{0}\bar{4}\bar{0}2$

$= (2,0,4)(\bar{2},\bar{0},\bar{4})\bar{4}0\underline{\bar{3}\bar{1}\bar{0}\bar{4}}\bar{0}2 = (2,0,4)(\bar{2},\bar{0},\bar{4})\bar{4}0(2,0,3,4,1)(\bar{2},\bar{0},\bar{3},\bar{4},\bar{1})0132\bar{0}2$

$= (2,3,4,0,1)(\bar{2},\bar{3},\bar{4},\bar{0},\bar{1})\bar{1}30132\underline{\bar{0}2} = (2,3,4,0,1)(\bar{2},\bar{3},\bar{4},\bar{0},\bar{1})\bar{1}30132(3,1,4)(\bar{3},\bar{1},\bar{4})\bar{2}0$

$= (1,2)(0,4)(\bar{1},\bar{2})(\bar{0},\bar{4})\bar{4}10412\bar{2}0 = (1,2)(0,4)(\bar{1},\bar{2})(\bar{0},\bar{4})\underline{\bar{4}1}0410$

$= (1,2)(0,4)(\bar{1},\bar{2})(\bar{0},\bar{4})(2,0,3)(\bar{2},\bar{0},\bar{3})\bar{1}40410 = (2,1,0,4,3)(\bar{2},\bar{1},\bar{0},\bar{4},\bar{3})\bar{1}40410$

$= (2,1,0,4,3)(\bar{2},\bar{1},\bar{0},\bar{4},\bar{3})\bar{1}\underline{4\bar{0}}\bar{0}410 = (2,1,0,4,3)(\bar{2},\bar{1},\bar{0},\bar{4},\bar{3})\bar{1}(2,1,3)(\bar{2},\bar{1},\bar{3})0\bar{4}\bar{0}410$

$= (2,3,1,0,4)(\bar{2},\bar{3},\bar{1},\bar{0},\bar{4})\bar{3}0\bar{4}\underline{\bar{0}4}10 = (2,3,1,0,4)(\bar{2},\bar{3},\bar{1},\bar{0},\bar{4})\bar{3}0\bar{4}(2,1,3)(\bar{2},\bar{1},\bar{3})\bar{4}010$

$= (1,0,4)(\bar{1},\bar{0},\bar{4})\bar{2}0\bar{4}\bar{4}010 = (1,0,4)(\bar{1},\bar{0},\bar{4})\bar{2}04010$

$= (1,0,4)(\bar{1},\bar{0},\bar{4})\bar{2}0\underline{\bar{0}4}010 = (1,0,4)(\bar{1},\bar{0},\bar{4})\bar{2}\bar{0}(2,1,3)(\bar{2},\bar{1},\bar{3})\bar{4}0010$

$= (2,1,0,4,3)(\bar{2},\bar{1},\bar{0},\bar{4},\bar{3})\bar{1}\bar{0}\bar{4}\underline{\bar{0}1}0 = (2,1,0,4,3)(\bar{2},\bar{1},\bar{0},\bar{4},\bar{3})\bar{1}\bar{0}\bar{4}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}00$

$= (2,1,0)(\bar{2},\bar{1},\bar{0})\bar{1}\bar{0}\bar{2}\bar{1}\bar{0} \in [01201].$ □

Lemma-56 ($012013 \in [01201]$)

Proof. $01\underline{2013} = 01(1,2,3,0,4)(\bar{1},\bar{2},\bar{3},\bar{0},\bar{4})\bar{1}\bar{0}\bar{2}\bar{4} = (1,2,3,0,4)(\bar{1},\bar{2},\bar{3},\bar{0},\bar{4})4\underline{2\bar{1}}\bar{0}\bar{2}\bar{4}$
$= (1,2,3,0,4)(\bar{1},\bar{2},\bar{3},\bar{0},\bar{4})4(3,4,0)(\bar{3},\bar{4},\bar{0})1\bar{2}\bar{0}\bar{2}\bar{4} = (1,2,4)(\bar{1},\bar{2},\bar{4})01\bar{2}\bar{0}\bar{2}\bar{4}$
$= (1,2,4)(\bar{1},\bar{2},\bar{4})01\underline{\bar{2}0}0\bar{2}\bar{4} = (1,2,4)(\bar{1},\bar{2},\bar{4})01(1,3,4)(\bar{1},\bar{3},\bar{4})\bar{0}20\bar{2}\bar{4}$
$= (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})03\bar{0}2\underline{0\bar{2}}\bar{4} = (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})03\bar{0}2(1,3,4)(\bar{1},\bar{3},\bar{4})2\bar{0}\bar{4}$
$= (1,2,3)(\bar{1},\bar{2},\bar{3})04\bar{0}22\bar{0}\bar{4} = (1,2,3)(\bar{1},\bar{2},\bar{3})0\underline{4\bar{0}}\bar{2}0\bar{4}$
$= (1,2,3)(\bar{1},\bar{2},\bar{3})0(1,3,2)(\bar{1},\bar{3},\bar{2})0\bar{4}\bar{2}\bar{0}\bar{4} = 00\bar{4}\bar{2}\bar{0}\bar{4} = \bar{0}\bar{4}\bar{2}\bar{0}\bar{4} \in [01201]$. $\square$

Lemma-57 ($012010 \in [01023]$)

Proof. $012010 = \bar{0}\underline{\bar{0}1}2010 = \bar{0}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}02010 = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}02010$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}02\bar{0}\underline{\bar{0}1}0 = (2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}\bar{1}02\bar{0}(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{1}00$
$= (2,4,3)(\bar{2},\bar{4},\bar{3})\bar{0}\bar{1}0\underline{3\bar{0}}\bar{1}\bar{0} = (2,4,3)(\bar{2},\bar{4},\bar{3})\bar{0}\bar{1}0(1,2,4)(\bar{1},\bar{2},\bar{4})0\bar{3}\bar{1}\bar{0}$
$= (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})\bar{0}\bar{2}00\bar{3}\bar{1}\bar{0} = (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})\bar{0}\underline{\bar{2}\bar{0}\bar{3}\bar{1}}\bar{0}$
$= (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})\bar{0}(3,2,1,0,4)(\bar{3},\bar{2},\bar{1},\bar{0},\bar{4})3024\bar{0} = (2,0,4)(\bar{2},\bar{0},\bar{4})\bar{4}3024\bar{0}$
$= (2,0,4)(\bar{2},\bar{0},\bar{4})4\underline{4302}4\bar{0} = (2,0,4)(\bar{2},\bar{0},\bar{4})4(0,4,2,3,1)(\bar{0},\bar{4},\bar{2},\bar{3},\bar{1})\bar{0}\bar{3}\bar{4}\bar{1}4\bar{0}$
$= (2,4,3,1,0)(\bar{2},\bar{4},\bar{3},\bar{1},\bar{0})2\bar{0}\bar{3}\bar{4}\underline{\bar{1}4}\bar{0} = (2,4,3,1,0)(\bar{2},\bar{4},\bar{3},\bar{1},\bar{0})2\bar{0}\bar{3}\bar{4}(2,3,0)(\bar{2},\bar{3},\bar{0})\bar{4}1\bar{0}$
$= (2,4,0,3,1)(\bar{2},\bar{4},\bar{0},\bar{3},\bar{1})3\bar{2}\bar{0}\bar{4}\bar{4}1\bar{0} = (2,4,0,3,1)(\bar{2},\bar{4},\bar{0},\bar{3},\bar{1})3\bar{2}\underline{\bar{0}4}1\bar{0}$
$= (2,4,0,3,1)(\bar{2},\bar{4},\bar{0},\bar{3},\bar{1})3\bar{2}(2,1,3)(\bar{2},\bar{1},\bar{3})\bar{4}01\bar{0} = (2,4,0)(\bar{2},\bar{4},\bar{0})2\bar{1}\bar{4}0\underline{1\bar{0}}$
$= (2,4,0)(\bar{2},\bar{4},\bar{0})2\bar{1}\bar{4}0(2,3,4)(\bar{2},\bar{3},\bar{4})\bar{0}0\bar{1} = (3,4,0)(\bar{3},\bar{4},\bar{0})3\bar{1}\bar{2}00\bar{1}$
$= (3,4,0)(\bar{3},\bar{4},\bar{0})\underline{3\bar{1}}\bar{2}\bar{0}\bar{1} = (3,4,0)(\bar{3},\bar{4},\bar{0})(2,0,4)(\bar{2},\bar{0},\bar{4})1\bar{3}\bar{2}\bar{0}\bar{1}$
$= (3,2,0)(\bar{3},\bar{2},\bar{0})1\underline{\bar{3}\bar{2}\bar{0}\bar{1}} = (3,2,0)(\bar{3},\bar{2},\bar{0})1(2,4,0,3,1)(\bar{2},\bar{4},\bar{0},\bar{3},\bar{1})0234$
$= (0,1,2,3,4)(\bar{0},\bar{1},\bar{2},\bar{3},\bar{4})20234 \in [01023]$. $\square$

Lemma-58 ($012043 \in [0121]$)

Proof. $01\underline{2043} = 01(2,3,0,1,4)(\bar{2},\bar{3},\bar{0},\bar{1},\bar{4})\bar{4}\bar{0}\bar{2}\bar{1} = (2,3,0,1,4)(\bar{2},\bar{3},\bar{0},\bar{1},\bar{4})14\bar{4}\bar{0}\bar{2}\bar{1}$
$= (2,3,0,1,4)(\bar{2},\bar{3},\bar{0},\bar{1},\bar{4})\underline{1\bar{0}}\bar{2}\bar{1} = (2,3,0,1,4)(\bar{2},\bar{3},\bar{0},\bar{1},\bar{4})(2,3,4)(\bar{2},\bar{3},\bar{4})0\bar{1}\bar{2}\bar{1}$
$= (2,4,3,0,1)(\bar{2},\bar{4},\bar{3},\bar{0},\bar{1})0\bar{1}\bar{2}\bar{1} = (2,4,3,0,1)(\bar{2},\bar{4},\bar{3},\bar{0},\bar{1})01\underline{1\bar{2}}\bar{1}$
$= (2,4,3,0,1)(\bar{2},\bar{4},\bar{3},\bar{0},\bar{1})01(0,4,3)(\bar{0},\bar{4},\bar{3})2\bar{1}\bar{1}$
$= (2,3,4,0,1)(\bar{2},\bar{3},\bar{4},\bar{0},\bar{1})4121 \in [0121]$. $\square$

Lemma-59 ($01204\bar{3} \in [01214]$)

Proof. $0\underline{1204}\bar{3} = 0(2,3,0,1,4)(\bar{2},\bar{3},\bar{0},\bar{1},\bar{4})\bar{0}\bar{2}\bar{1}\bar{3}\bar{3} = (2,3,0,1,4)(\bar{2},\bar{3},\bar{0},\bar{1},\bar{4})\underline{1\bar{0}}\bar{2}\bar{1}3$
$= (2,3,0,1,4)(\bar{2},\bar{3},\bar{0},\bar{1},\bar{4})(2,3,4)(\bar{2},\bar{3},\bar{4})0\bar{1}\bar{2}\bar{1}3 = (2,4,3,0,1)(\bar{2},\bar{4},\bar{3},\bar{0},\bar{1})01\underline{1\bar{2}}\bar{1}3$
$= (2,4,3,0,1)(\bar{2},\bar{4},\bar{3},\bar{0},\bar{1})01(0,4,3)(\bar{0},\bar{4},\bar{3})2\bar{1}\bar{1}3 = (1,2,0)(\bar{1},\bar{2},\bar{0})41213 \in [01214]$. $\square$

Lemma-60 ($012041 \in [0123]$)

Proof. $01\underline{2041} = 01(4,2,1,0,3)(\bar{4},\bar{2},\bar{1},\bar{0},\bar{3})\bar{4}\bar{0}\bar{2}\bar{3} = (4,2,1,0,3)(\bar{4},\bar{2},\bar{1},\bar{0},\bar{3})3\underline{0\bar{4}}\bar{0}\bar{2}\bar{3}$
$= (4,2,1,0,3)(\bar{4},\bar{2},\bar{1},\bar{0},\bar{3})3(1,2,3)(\bar{1},\bar{2},\bar{3})4\bar{0}\bar{0}\bar{2}\bar{3} = (3,4)(1,0)(\bar{3},\bar{4})(\bar{1},\bar{0})140\bar{2}\bar{3}$
$= (3,4)(1,0)(\bar{3},\bar{4})(\bar{1},\bar{0})\underline{140\bar{2}}2\bar{3} = (3,4)(1,0)(\bar{3},\bar{4})(\bar{1},\bar{0})(2,4,3,0,1)(\bar{2},\bar{4},\bar{3},\bar{0},\bar{1})\bar{0}\bar{4}\bar{1}\underline{\bar{3}2}\bar{3}$
$= (4,0,2)(\bar{4},\bar{0},\bar{2})\bar{0}\bar{4}\bar{1}(0,4,1)(\bar{0},\bar{4},\bar{1})\bar{2}3\bar{3} = (1,0,2)(\bar{1},\bar{0},\bar{2})\bar{4}\bar{1}\bar{0}\bar{2} \in [0123].$ $\square$

Lemma-61 ($01204\bar{1} \in [01203]$)

Proof. $0120\underline{4\bar{1}} = 0120(3,0,2)(\bar{3},\bar{0},\bar{2})1\bar{4} = (3,0,2)(\bar{3},\bar{0},\bar{2})21321\bar{4}$
$= (3,0,2)(\bar{3},\bar{0},\bar{2})\bar{2}\underline{\bar{2}1}321\bar{4} = (3,0,2)(\bar{3},\bar{0},\bar{2})\bar{2}(3,0,4)(\bar{3},\bar{0},\bar{4})\bar{1}2321\bar{4}$
$= (3,4)(2,0)(\bar{3},\bar{4})(\bar{2},\bar{0})\bar{2}\bar{1}2321\bar{4} = (3,4)(2,0)(\bar{3},\bar{4})(\bar{2},\bar{0})\bar{2}\bar{1}\bar{2}\underline{23}21\bar{4}$
$= (3,4)(2,0)(\bar{3},\bar{4})(\bar{2},\bar{0})\bar{2}\bar{1}\bar{2}(0,1,4)(\bar{0},\bar{1},\bar{4})\bar{3}221\bar{4} = (0,2,1,4,3)(\bar{0},\bar{2},\bar{1},\bar{4},\bar{3})\bar{2}\bar{4}\bar{2}3\underline{21}\bar{4}$
$= (0,2,1,4,3)(\bar{0},\bar{2},\bar{1},\bar{4},\bar{3})\bar{2}\bar{4}\bar{2}\bar{3}(3,0,4)(\bar{3},\bar{0},\bar{4})\bar{1}2\bar{4} = (1,3,4,0,2)(\bar{1},\bar{3},\bar{4},\bar{0},\bar{2})2\underline{\bar{3}\bar{2}\bar{0}\bar{1}}2\bar{4}$
$= (1,3,4,0,2)(\bar{1},\bar{3},\bar{4},\bar{0},\bar{2})\bar{2}(0,3,1,2,4)(\bar{0},\bar{3},\bar{1},\bar{2},\bar{4})02342\bar{4} = (3,0,4)(\bar{3},\bar{0},\bar{4})\bar{4}0234\underline{2\bar{4}}$
$= (3,0,4)(\bar{3},\bar{0},\bar{4})\bar{4}0234(3,0,1)(\bar{3},\bar{0},\bar{1})4\bar{2} = (3,1)(4,0)(\bar{3},\bar{1})(\bar{4},\bar{0})\bar{4}12044\bar{2}$
$= (3,1)(4,0)(\bar{3},\bar{1})(\bar{4},\bar{0})\bar{4}120\bar{4}\bar{2} = (3,1)(4,0)(\bar{3},\bar{1})(\bar{4},\bar{0})4\underline{4120}\bar{4}\bar{2}$
$= (3,1)(4,0)(\bar{3},\bar{1})(\bar{4},\bar{0})4(2,4,0,1,3)(\bar{2},\bar{4},\bar{0},\bar{1},\bar{3})\bar{2}\bar{1}\bar{4}\bar{3}4\bar{2}$
$= (1,2,4)(\bar{1},\bar{2},\bar{4})\underline{0\bar{2}}\bar{1}\bar{4}\bar{3}4\bar{2} = (1,2,4)(\bar{1},\bar{2},\bar{4})(1,3,4)(\bar{1},\bar{3},\bar{4})2\bar{0}\bar{1}\bar{4}\bar{3}4\bar{2}$
$= (3,4)(2,1)(\bar{3},\bar{4})(\bar{2},\bar{1})2\underline{\bar{0}\bar{1}\bar{4}\bar{3}}4\bar{2} = (3,4)(2,1)(\bar{3},\bar{4})(\bar{2},\bar{1})2(1,2,4,0,3)(\bar{1},\bar{2},\bar{4},\bar{0},\bar{3})4102\bar{4}\bar{2}$
$= (1,4)(3,0)(\bar{1},\bar{4})(\bar{3},\bar{0})44102\bar{4}\bar{2} = (1,4)(3,0)(\bar{1},\bar{4})(\bar{3},\bar{0})\bar{4}10\underline{2\bar{4}}\bar{2}$
$= (1,4)(3,0)(\bar{1},\bar{4})(\bar{3},\bar{0})\bar{4}10(1,3,0)(\bar{1},\bar{3},\bar{0})42\bar{2} = (3,1,4)(\bar{3},\bar{1},\bar{4})\bar{4}\underline{3142}$
$= (3,1,4)(\bar{3},\bar{1},\bar{4})\bar{4}(3,2,1,0,4)(\bar{3},\bar{2},\bar{1},\bar{0},\bar{4})\bar{4}\bar{1}3\bar{0}$
$= (1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2})\bar{3}\bar{4}\bar{1}3\bar{0} \in [01203].$ $\square$

Lemma-62 ($0120\bar{3}0 \in [0123]$)

Proof. $0120\underline{\bar{3}0} = 0120(1,4,2)(\bar{1},\bar{4},\bar{2})\bar{0}3 = (1,4,2)(\bar{1},\bar{4},\bar{2})0410\bar{0}3$
$= (1,4,2)(\bar{1},\bar{4},\bar{2})0413 \in [0123].$ $\square$

Lemma-63 ($0120\bar{3}1 \in [0120\bar{3}]$)

Proof. $0120\bar{3}1 = 0\bar{1}\underline{\bar{1}2}0\bar{3}1 = 0\bar{1}(3,4,0)(\bar{3},\bar{4},\bar{0})\bar{2}10\bar{3}1 = (3,4,0)(\bar{3},\bar{4},\bar{0})3\bar{1}\bar{2}10\underline{\bar{3}1}$
$= (3,4,0)(\bar{3},\bar{4},\bar{0})3\bar{1}\bar{2}10(2,4,0)(\bar{2},\bar{4},\bar{0})\bar{1}3 = (3,0)(4,2)(\bar{3},\bar{0})(\bar{4},\bar{2})3\bar{1}41\underline{2\bar{1}}3$
$= (3,0)(4,2)(\bar{3},\bar{0})(\bar{4},\bar{2})3\bar{1}41(3,4,0)(\bar{3},\bar{4},\bar{0})1\bar{2}3 = (0,4,2)(\bar{0},\bar{4},\bar{2})4\bar{1}\bar{0}11\bar{2}3$
$= (0,4,2)(\bar{0},\bar{4},\bar{2})\underline{4\bar{1}}\bar{0}\bar{1}\bar{2}3 = (0,4,2)(\bar{0},\bar{4},\bar{2})(3,2,0)(\bar{3},\bar{2},\bar{0})1\bar{4}\bar{0}\bar{1}\bar{2}3$

$= (0,4)(2,3)(\bar{0},\bar{4})(\bar{2},\bar{3})1\underline{\bar{4}\bar{0}\bar{1}\bar{2}}3 = (0,4)(2,3)(\bar{0},\bar{4})(\bar{2},\bar{3})1(4,2,0,3,1)(\bar{4},\bar{2},\bar{0},\bar{3},\bar{1})10433$
$= (2,0,1,4,3)(\bar{2},\bar{0},\bar{1},\bar{4},\bar{3})4104\bar{3} \in [0120\bar{3}].$ □

Lemma-64 ($0120\bar{3}2 \in [0123]$)

Proof. $0120\underline{\bar{3}2} = 0120(0,4,1)(\bar{0},\bar{4},\bar{1})\bar{2}3 = (0,4,1)(\bar{0},\bar{4},\bar{1})402\underline{4\bar{2}}3$
$= (0,4,1)(\bar{0},\bar{4},\bar{1})402(3,1,0)(\bar{3},\bar{1},\bar{0})2\bar{4}3 = (0,4)(1,3)(\bar{0},\bar{4})(\bar{1},\bar{3})4322\bar{4}3$
$= (0,4)(1,3)(\bar{0},\bar{4})(\bar{1},\bar{3})4\underline{3\bar{2}}\bar{4}3 = (0,4)(1,3)(\bar{0},\bar{4})(\bar{1},\bar{3})4(1,4,0)(\bar{1},\bar{4},\bar{0})2\bar{3}\bar{4}3$
$= (3,4,1)(\bar{3},\bar{4},\bar{1})02\bar{3}\bar{4}3 = (3,4,1)(\bar{3},\bar{4},\bar{1})023\underline{3\bar{4}}3$
$= (3,4,1)(\bar{3},\bar{4},\bar{1})023(0,2,1)(\bar{0},\bar{2},\bar{1})4\bar{3}3 = (4,0,2,1,3)(\bar{4},\bar{0},\bar{2},\bar{1},\bar{3})\underline{2134}$
$= (4,0,2,1,3)(\bar{4},\bar{0},\bar{2},\bar{1},\bar{3})(2,4,1,0,3)(\bar{2},\bar{4},\bar{1},\bar{0},\bar{3})\bar{3}\bar{1}\bar{2}\bar{0}$
$= (4,3,1,2,0)(\bar{4},\bar{3},\bar{1},\bar{2},\bar{0})\bar{3}\bar{1}\bar{2}\bar{0} \in [0123].$ □

Lemma-65 ($0120\bar{3}4 \in [0120]$)

Proof. $0120\bar{3}4 = 0\underline{1203}34 = 0(0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})\bar{0}\bar{2}\bar{1}\bar{4}34$
$= (0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})1\bar{0}\bar{2}\bar{1}\underline{\bar{4}3}4 = (0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})1\bar{0}\bar{2}\bar{1}(0,2,1)(\bar{0},\bar{2},\bar{1})\bar{3}44$
$= (2,4)(1,3)(\bar{2},\bar{4})(\bar{1},\bar{3})0\bar{2}\underline{\bar{1}\bar{0}\bar{3}\bar{4}} = (2,4)(1,3)(\bar{2},\bar{4})(\bar{1},\bar{3})0\bar{2}(3,1,4,0,2)(\bar{3},\bar{1},\bar{4},\bar{0},\bar{2})3012$
$= (3,4)(2,0)(\bar{3},\bar{4})(\bar{2},\bar{0})2\bar{3}3012 = (3,4)(2,0)(\bar{3},\bar{4})(\bar{2},\bar{0})2012 \in [0120].$ □

Lemma-66 ($0120\bar{3}\bar{0} \in [01\bar{2}\bar{0}]$)

Proof. $0120\bar{3}\bar{0} = 0120\underline{\bar{3}0}0 = 0120(1,4,2)(\bar{1},\bar{4},\bar{2})\bar{0}30 = (1,4,2)(\bar{1},\bar{4},\bar{2})0410\bar{0}30$
$= (1,4,2)(\bar{1},\bar{4},\bar{2})\underline{0413}0 = (1,4,2)(\bar{1},\bar{4},\bar{2})(4,2,1,0,3)(\bar{4},\bar{2},\bar{1},\bar{0},\bar{3})\bar{1}\bar{4}\bar{0}\bar{2}0$
$= (1,2,0,3,4)(\bar{1},\bar{2},\bar{0},\bar{3},\bar{4})\bar{1}\bar{4}\bar{0}\bar{2}0 = (1,2,0,3,4)(\bar{1},\bar{2},\bar{0},\bar{3},\bar{4})\bar{1}\bar{4}0\underline{0\bar{2}}0$
$= (1,2,0,3,4)(\bar{1},\bar{2},\bar{0},\bar{3},\bar{4})\bar{1}\bar{4}0(1,3,4)(\bar{1},\bar{3},\bar{4})2\bar{0}0 = (1,2,0,4,3)(\bar{1},\bar{2},\bar{0},\bar{4},\bar{3})\bar{3}\underline{\bar{1}0}2$
$= (1,2,0,4,3)(\bar{1},\bar{2},\bar{0},\bar{4},\bar{3})\bar{3}(2,4,3)(\bar{2},\bar{4},\bar{3})\bar{0}12$
$= (1,4,2,0,3)(\bar{1},\bar{4},\bar{2},\bar{0},\bar{3})\bar{2}\bar{0}12 \in [01\bar{2}\bar{0}].$ □

Lemma-67 ($0120\bar{3}\bar{1} \in [0120\bar{3}]$)

Proof. $0120\bar{3}\bar{1} = 01203\underline{3\bar{1}} = 01203(2,0,4)(\bar{2},\bar{0},\bar{4})1\bar{3} = (2,0,4)(\bar{2},\bar{0},\bar{4})4\underline{1043}1\bar{3}$
$= (2,0,4)(\bar{2},\bar{0},\bar{4})4(4,1,3,0,2)(\bar{4},\bar{1},\bar{3},\bar{0},\bar{2})\bar{4}\bar{0}\bar{1}\bar{2}1\bar{3} = (0,1,3)(\bar{0},\bar{1},\bar{3})1\bar{4}\bar{0}\bar{1}\bar{2}1\bar{3}$
$= (0,1,3)(\bar{0},\bar{1},\bar{3})1\bar{4}\bar{0}1\underline{1\bar{2}}1\bar{3} = (0,1,3)(\bar{0},\bar{1},\bar{3})1\bar{4}\bar{0}1(0,4,3)(\bar{0},\bar{4},\bar{3})2\bar{1}1\bar{3}$
$= (0,1)(3,4)(\bar{0},\bar{1})(\bar{3},\bar{4})\underline{1\bar{3}}\bar{4}12\bar{3} = (0,1)(3,4)(\bar{0},\bar{1})(\bar{3},\bar{4})(2,4,0)(\bar{2},\bar{4},\bar{0})3\bar{1}\bar{4}12\bar{3}$
$= (0,1,2,4,3)(\bar{0},\bar{1},\bar{2},\bar{4},\bar{3})31\underline{1\bar{4}}12\bar{3} = (0,1,2,4,3)(\bar{0},\bar{1},\bar{2},\bar{4},\bar{3})31(2,0,3)(\bar{2},\bar{0},\bar{3})4\bar{1}12\bar{3}$
$= (0,1)(2,4)(\bar{0},\bar{1})(\bar{2},\bar{4})2142\bar{3} \in [0120\bar{3}].$ □

Lemma-68 $(0120\bar{3}\bar{2} \in [01\bar{2}\bar{3}])$

Proof. $0120\bar{3}\bar{2} = 01\bar{2}\underline{\bar{2}0}\bar{3}\bar{2} = 01\bar{2}(4,1,3)(\bar{4},\bar{1},\bar{3})\bar{0}2\bar{3}\bar{2} = (4,1,3)(\bar{4},\bar{1},\bar{3})03\bar{2}\bar{0}\underline{2\bar{3}}\bar{2}$
$= (4,1,3)(\bar{4},\bar{1},\bar{3})03\bar{2}\bar{0}(4,1,0)(\bar{4},\bar{1},\bar{0})3\bar{2}\bar{2} = (1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0})4\underline{3\bar{2}}\bar{4}32$
$= (1,3)(4,0)(\bar{1},\bar{3})(\bar{4},\bar{0})4(1,4,0)(\bar{1},\bar{4},\bar{0})2\bar{3}\bar{4}32 = (4,1,3)(\bar{4},\bar{1},\bar{3})02\bar{3}\bar{4}32$
$= (4,1,3)(\bar{4},\bar{1},\bar{3})023\underline{3\bar{4}}32 = (4,1,3)(\bar{4},\bar{1},\bar{3})023(2,1,0)(\bar{2},\bar{1},\bar{0})4\bar{3}32$
$= (1,3,4,0,2)(\bar{1},\bar{3},\bar{4},\bar{0},\bar{2})2\underline{1342}$
$= (1,3,4,0,2)(\bar{1},\bar{3},\bar{4},\bar{0},\bar{2})2(4,1,2,3,0)(\bar{4},\bar{1},\bar{2},\bar{3},\bar{0})\bar{4}\bar{3}\bar{1}\bar{0}$
$= (1,0,3)(\bar{1},\bar{0},\bar{3})\underline{3\bar{4}}\bar{3}\bar{1}\bar{0} = (1,0,3)(\bar{1},\bar{0},\bar{3})(1,0,2)(\bar{1},\bar{0},\bar{2})4\bar{3}\bar{3}\bar{1}\bar{0}$
$= (1,2)(0,3)(\bar{1},\bar{2})(\bar{0},\bar{3})43\bar{1}\bar{0} \in [01\bar{2}\bar{3}].$ □

Lemma-69 $(0120\bar{3}\bar{4} \in [01204])$

Proof. $0120\bar{3}\bar{4} = 01203\underline{3\bar{4}} = 01203(1,0,2)(\bar{1},\bar{0},\bar{2})4\bar{3} = (1,0,2)(\bar{1},\bar{0},\bar{2})2\underline{0123}4\bar{3}$
$= (1,0,2)(\bar{1},\bar{0},\bar{2})2(2,0,3,1,4)(\bar{1},\bar{3},\bar{4},\bar{0},\bar{2})\bar{2}\bar{1}\bar{0}\bar{4}4\bar{3} = (4,2)(1,3)(\bar{4},\bar{2})(\bar{1},\bar{3})0\underline{\bar{2}\bar{1}\bar{0}\bar{3}}$
$= (4,2)(1,3)(\bar{4},\bar{2})(\bar{1},\bar{3})0(0,2,3,1,4)(\bar{0},\bar{2},\bar{3},\bar{1},\bar{4})0124$
$= (0,2)(4,3)(\bar{0},\bar{2})(\bar{4},\bar{3})20124 \in [01204].$ □

Lemma-70 $(012141 \in [01023])$

Proof. $012141 = 0121\underline{4\bar{1}}\bar{1} = 0121(3,0,2)(\bar{3},\bar{0},\bar{2})1\bar{4}\bar{1} = (3,0,2)(\bar{3},\bar{0},\bar{2})21311\bar{4}\bar{1}$
$= (3,0,2)(\bar{3},\bar{0},\bar{2})21\underline{3\bar{1}}\bar{4}\bar{1} = (3,0,2)(\bar{3},\bar{0},\bar{2})21(2,0,4)(\bar{2},\bar{0},\bar{4})1\bar{3}\bar{4}\bar{1}$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})011\bar{3}\bar{4}\bar{1} = (2,3,4)(\bar{2},\bar{3},\bar{4})\underline{0\bar{1}}\bar{3}\bar{4}\bar{1}$
$= (2,3,4)(\bar{2},\bar{3},\bar{4})(2,4,3)(\bar{2},\bar{4},\bar{3})1\bar{0}\bar{3}\bar{4}\bar{1} = 1\underline{\bar{0}\bar{3}\bar{4}\bar{1}}$
$= 1(0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})4302 = (0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})34302$
$= (0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})\underline{3\bar{4}}\bar{4}302 = (0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})(1,0,2)(\bar{1},\bar{0},\bar{2})4\bar{3}\bar{4}302$
$= (4,2)(1,3)(\bar{4},\bar{2})(\bar{1},\bar{3})43\underline{3\bar{4}}302 = (4,2)(1,3)(\bar{4},\bar{2})(\bar{1},\bar{3})43(1,0,2)(\bar{1},\bar{0},\bar{2})4\bar{3}302$
$= (1,3,0,2,4)(\bar{1},\bar{3},\bar{0},\bar{2},\bar{4})43402 \in [01023].$ □

Lemma-71 $(01214\bar{1} \in [0\bar{1}\bar{2}\bar{3}])$

Proof. $0121\underline{4\bar{1}} = 0121(3,0,2)(\bar{3},\bar{0},\bar{2})1\bar{4} = (3,0,2)(\bar{3},\bar{0},\bar{2})21311\bar{4} = (3,0,2)(\bar{3},\bar{0},\bar{2})21\underline{3\bar{1}}\bar{4}$
$= (3,0,2)(\bar{3},\bar{0},\bar{2})21(2,0,4)(\bar{2},\bar{0},\bar{4})1\bar{3}\bar{4} = (3,4,2)(\bar{3},\bar{4},\bar{2})011\bar{3}\bar{4}$
$= (3,4,2)(\bar{3},\bar{4},\bar{2})0\bar{1}\bar{3}\bar{4} \in [0\bar{1}\bar{2}\bar{3}].$ □

Lemma-72 $(012140 \in [01203])$

Proof. $01\underline{2140} = 01(2,0,1,3,4)(\bar2,\bar0,\bar1,\bar3,\bar4)\bar4\bar1\bar2\bar3 = (2,0,1,3,4)(\bar2,\bar0,\bar1,\bar3,\bar4)13\bar4\bar1\bar2\bar3$
$= (2,0,1,3,4)(\bar2,\bar0,\bar1,\bar3,\bar4)1\bar3\underline{\bar3\bar4\bar1\bar2}\bar3$
$= (2,0,1,3,4)(\bar2,\bar0,\bar1,\bar3,\bar4)1\bar3(0,1,3,2,4)(\bar0,\bar1,\bar3,\bar2,\bar4)1430\bar3$
$= (1,2)(0,3)(\bar1,\bar2)(\bar0,\bar3)3\bar2143\underline{0\bar3} = (1,2)(0,3)(\bar1,\bar2)(\bar0,\bar3)3\bar2143(1,4,2)(\bar1,\bar4,\bar2)3\bar0$
$= (4,2)(0,3)(\bar4,\bar2)(\bar0,\bar3)3\bar14233\bar0 = (4,2)(0,3)(\bar4,\bar2)(\bar0,\bar3)\underline{3\bar1}42\bar3\bar0$
$= (4,2)(0,3)(\bar4,\bar2)(\bar0,\bar3)(0,4,2)(\bar0,\bar4,\bar2)1\bar342\bar3\bar0 = (3,4,0)(\bar3,\bar4,\bar0)1\underline{\bar34}2\bar3\bar0$
$= (3,4,0)(\bar3,\bar4,\bar0)1(2,0,1)(\bar2,\bar0,\bar1)\bar432\bar3\bar0 = (4,1,2,0,3)(\bar4,\bar1,\bar2,\bar0,\bar3)2\bar43\underline{2\bar3}\bar0$
$= (4,1,2,0,3)(\bar4,\bar1,\bar2,\bar0,\bar3)2\bar43(0,4,1)(\bar0,\bar4,\bar1)3\bar2\bar0 = (4,0,3,1,2)(\bar4,\bar0,\bar3,\bar1,\bar2)2\bar133\bar2\bar0$
$= (4,0,3,1,2)(\bar4,\bar0,\bar3,\bar1,\bar2)2\underline{\bar1\bar3\bar2\bar0}$
$= (4,0,3,1,2)(\bar4,\bar0,\bar3,\bar1,\bar2)2(1,0,3,4,2)(\bar1,\bar0,\bar3,\bar4,\bar2)2314$
$= (4,3,0)(\bar4,\bar3,\bar0)12314 \in [01203].$ □

Lemma-73 ($01214\bar0 \in [0121]$)

Proof. $01214\bar0 = 01\underline{2140}0 = 01(2,0,1,3,4)(\bar2,\bar0,\bar1,\bar3,\bar4)\bar4\bar1\bar2\bar30$
$= (2,0,1,3,4)(\bar2,\bar0,\bar1,\bar3,\bar4)13\bar4\bar1\bar2\underline{\bar30} = (2,0,1,3,4)(\bar2,\bar0,\bar1,\bar3,\bar4)13\bar4\bar1\bar2(1,4,2)(\bar1,\bar4,\bar2)\bar03$
$= (0,4,1,3,2)(\bar0,\bar4,\bar1,\bar3,\bar2)43\underline{\bar2\bar4\bar1\bar0}3$
$= (0,4,1,3,2)(\bar0,\bar4,\bar1,\bar3,\bar2)43(0,4,3,1,2)(\bar0,\bar4,\bar3,\bar1,\bar2)14233$
$= (4,2)(0,3)(\bar4,\bar2)(\bar0,\bar3)31142\bar3 = (4,2)(0,3)(\bar4,\bar2)(\bar0,\bar3)\underline{3\bar1}42\bar3$
$= (4,2)(0,3)(\bar4,\bar2)(\bar0,\bar3)(4,2,0)(\bar4,\bar2,\bar0)1\bar342\bar3 = (3,4,0)(\bar3,\bar4,\bar0)1\underline{\bar34}2\bar3$
$= (3,4,0)(\bar3,\bar4,\bar0)1(1,2,0)(\bar1,\bar2,\bar0)\bar432\bar3 = (1,2,0,3,4)(\bar1,\bar2,\bar0,\bar3,\bar4)2\bar43\underline{2\bar3}$
$= (1,2,0,3,4)(\bar1,\bar2,\bar0,\bar3,\bar4)2\bar43(4,1,0)(\bar4,\bar1,\bar0)3\bar2 = (0,3,1,2,4)(\bar0,\bar3,\bar1,\bar2,\bar4)2\bar133\bar2$
$= (0,3,1,2,4)(\bar0,\bar3,\bar1,\bar2,\bar4)\underline{2\bar1}\bar3\bar2 = (0,3,1,2,4)(\bar0,\bar3,\bar1,\bar2,\bar4)(3,4,0)(\bar3,\bar4,\bar0)1\bar2\bar3\bar2$
$= (4,3,1,2,0)(\bar4,\bar3,\bar1,\bar2,\bar0)1\bar2\bar3\bar2 = (4,3,1,2,0)(\bar4,\bar3,\bar1,\bar2,\bar0)12\underline{\bar2\bar3}\bar2$
$= (4,3,1,2,0)(\bar4,\bar3,\bar1,\bar2,\bar0)12(4,1,0)(\bar4,\bar1,\bar0)3\bar2\bar2$
$= (1,2,4,3,0)(\bar1,\bar2,\bar4,\bar3,\bar0)0232 \in [0121].$ □

Lemma-74 ($012030 \in [012\bar3\bar0]$)

Proof. $012030 = \bar0\underline{01}2030 = \bar0(3,4,2)(\bar3,\bar4,\bar2)\bar102030 = (3,4,2)(\bar3,\bar4,\bar2)\bar0\bar102030$
$= (3,4,2)(\bar3,\bar4,\bar2)\bar0\bar1\bar0\underline{02}030 = (3,4,2)(\bar3,\bar4,\bar2)\bar0\bar1\bar0(1,4,3)(\bar1,\bar4,\bar3)\bar20030$
$= (1,4,2)(\bar1,\bar4,\bar2)\bar0\bar40\bar2\underline{03}0 = (1,4,2)(\bar1,\bar4,\bar2)\bar0\bar40\bar2(1,2,4)(\bar1,\bar2,\bar4)\bar300$
$= \bar0\bar1\bar0\bar43\bar0 = \underline{\bar01}1\bar0\bar43\bar0 = (3,4,2)(\bar3,\bar4,\bar2)\bar10\underline{1\bar0}\bar43\bar0$
$= (3,4,2)(\bar3,\bar4,\bar2)\bar10(3,4,2)(\bar3,\bar4,\bar2)0\bar1\bar43\bar0 = (4,3,2)(\bar4,\bar3,\bar2)\bar100\bar1\bar43\bar0$
$= (4,3,2)(\bar4,\bar3,\bar2)\bar10\underline{\bar1\bar43\bar0} = (4,3,2)(\bar4,\bar3,\bar2)\bar1\bar0(1,0,4,2,3)(\bar1,\bar0,\bar4,\bar2,\bar3)3412$

$= (4,1,0)(\bar{4},\bar{1},\bar{0})\bar{0}\underline{\bar{4}3}412 = (4,1,0)(\bar{4},\bar{1},\bar{0})\bar{0}(2,1,0)(\bar{2},\bar{1},\bar{0})\bar{3}4412$
$= (1,2)(0,4)(\bar{1},\bar{2})(\bar{0},\bar{4})\bar{2}\bar{3}\bar{4}412 \in [012\bar{3}\bar{0}]$. □

Lemma-75 $(012031 \in [0120])$

Proof. $0\underline{1203}1 = 0(0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})\bar{0}\bar{2}\bar{1}\bar{4}1 = (0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})\underline{1\bar{0}}\bar{2}\bar{1}\bar{4}1$
$= (0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})(3,4,2)(\bar{3},\bar{4},\bar{2})0\bar{1}\bar{2}\bar{1}\bar{4}1 = (0,1,4)(\bar{0},\bar{1},\bar{4})01\underline{1\bar{2}}\bar{1}\bar{4}1$
$= (0,1,4)(\bar{0},\bar{1},\bar{4})01(4,3,0)(\bar{4},\bar{3},\bar{0})2\bar{1}\bar{1}\bar{4}1 = (0,1,3)(\bar{0},\bar{1},\bar{3})412\underline{1\bar{4}}1$
$= (0,1,3)(\bar{0},\bar{1},\bar{3})412(0,3,2)(\bar{0},\bar{3},\bar{2})4\bar{1}1 = (0,1,2)(\bar{0},\bar{1},\bar{2})4104 \in [0120]$. □

Lemma-76 $(012032 \in [0123])$

Proof. $0\underline{1203}2 = 0(0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})\bar{0}\bar{2}\bar{1}\bar{4}2 = (0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})1\bar{0}\bar{2}\bar{1}\underline{\bar{4}2}$
$= (0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})1\bar{0}\bar{2}\bar{1}(0,1,3)(\bar{0},\bar{1},\bar{3})\bar{2}4 = (0,3,2,4,1)(\bar{0},\bar{3},\bar{2},\bar{4},\bar{1})\underline{3\bar{1}}\bar{2}\bar{3}\bar{2}4$
$= (0,3,2,4,1)(\bar{0},\bar{3},\bar{2},\bar{4},\bar{1})(0,4,2)(\bar{0},\bar{4},\bar{2})1\bar{3}\bar{2}\bar{3}\bar{2}4 = (4,1)(0,3)(\bar{4},\bar{1})(\bar{0},\bar{3})1\bar{3}\bar{2}\bar{3}\bar{2}4$
$= (4,1)(0,3)(\bar{4},\bar{1})(\bar{0},\bar{3})13\underline{3\bar{2}}\bar{3}\bar{2}4 = (4,1)(0,3)(\bar{4},\bar{1})(\bar{0},\bar{3})13(0,1,4)(\bar{0},\bar{1},\bar{4})2\bar{3}\bar{3}\bar{2}4$
$= (0,3,1)(\bar{0},\bar{3},\bar{1})4323\bar{2}4 = (0,3,1)(\bar{0},\bar{3},\bar{1})43\bar{2}\underline{\bar{2}3}\bar{2}4$
$= (0,3,1)(\bar{0},\bar{3},\bar{1})43\bar{2}(0,1,4)(\bar{0},\bar{1},\bar{4})\bar{3}2\bar{2} = (0,3,4)(\bar{0},\bar{3},\bar{4})0\underline{3\bar{2}}\bar{3}4$
$= (0,3,4)(\bar{0},\bar{3},\bar{4})0(0,1,4)(\bar{0},\bar{1},\bar{4})2\bar{3}\bar{3}4 = (4,1)(0,3)(\bar{4},\bar{1})(\bar{0},\bar{3})1234 \in [0123]$. □

Lemma-77 $(012034 \in [0121])$

Proof. $0\underline{1203}4 = 0(0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})\bar{0}\bar{2}\bar{1}\bar{4}4 = (0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})\underline{1\bar{0}}\bar{2}\bar{1}$
$= (0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})(3,4,2)(\bar{3},\bar{4},\bar{2})0\bar{1}\bar{2}\bar{1} = (0,1,4)(\bar{0},\bar{1},\bar{4})01\underline{1\bar{2}}\bar{1}$
$= (0,1,4)(\bar{0},\bar{1},\bar{4})01(0,4,3)(\bar{0},\bar{4},\bar{3})2\bar{1}\bar{1} = (0,1,3)(\bar{0},\bar{1},\bar{3})4121 \in [0121]$. □

Lemma-78 $(01203\bar{0} \in [01023])$

Proof. $01203\bar{0} = \bar{0}\underline{\bar{0}1}203\bar{0} = \bar{0}(3,4,2)(\bar{3},\bar{4},\bar{2})\bar{1}0203\bar{0} = (3,4,2)(\bar{3},\bar{4},\bar{2})\bar{0}\bar{1}0203\bar{0}$
$= (3,4,2)(\bar{3},\bar{4},\bar{2})\bar{0}\bar{1}\bar{0}\underline{\bar{0}2}03\bar{0} = (3,4,2)(\bar{3},\bar{4},\bar{2})\bar{0}\bar{1}\bar{0}(3,1,4)(\bar{3},\bar{1},\bar{4})\bar{2}003\bar{0}$
$= (2,1,4)(\bar{2},\bar{1},\bar{4})\bar{0}\bar{4}\bar{0}\bar{2}\underline{\bar{0}3}\bar{0} = (2,1,4)(\bar{2},\bar{1},\bar{4})\bar{0}\bar{4}\bar{0}\bar{2}(1,2,4)(\bar{1},\bar{2},\bar{4})\bar{3}0\bar{0}$
$= \bar{0}\bar{1}\bar{0}\bar{4}\bar{3} \in [01023]$. □

Lemma-79 $(01203\bar{1} \in [01201])$

Proof. $0\underline{1203}\bar{1} = 0(0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})\bar{0}\bar{2}\bar{1}\bar{4}\bar{1} = (0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})\underline{1\bar{0}}\bar{2}\bar{1}\bar{4}\bar{1}$
$= (0,1,3,2,4)(\bar{0},\bar{1},\bar{3},\bar{2},\bar{4})(4,2,3)(\bar{4},\bar{2},\bar{3})0\bar{1}\bar{2}\bar{1}\bar{4}\bar{1} = (0,1,4)(\bar{0},\bar{1},\bar{4})01\underline{1\bar{2}}\bar{1}\bar{4}\bar{1}$

$= (0,1,4)(\bar0,\bar1,\bar4)01(0,4,3)(\bar0,\bar4,\bar3)2\bar1\bar1\bar4\bar1 = (0,1,3)(\bar0,\bar1,\bar3)412\underline{1\bar4}\bar1$
$= (0,1,3)(\bar0,\bar1,\bar3)412(2,0,3)(\bar2,\bar0,\bar3)4\bar1\bar1 = (0,1,2)(\bar0,\bar1,\bar2)41041 \in [01201].$ $\square$

Lemma-80 ($01203\bar2 \in [01204]$)

Proof. $01203\bar2 = 01\bar2\underline{\bar2 0}3\bar2 = 01\bar2(4,1,3)(\bar4,\bar1,\bar3)\bar0 23\bar2 = (4,1,3)(\bar4,\bar1,\bar3)03\bar2\bar0 23\bar2$
$= (4,1,3)(\bar4,\bar1,\bar3)03\bar2\bar0\bar2\underline{\bar2 3}\bar2 = (4,1,3)(\bar4,\bar1,\bar3)03\bar2\bar0\bar2(0,1,4)(\bar0,\bar1,\bar4)\bar3 2\bar2$
$= (0,1,3)(\bar0,\bar1,\bar3)13\bar2\bar1\bar2\bar3 = (0,1,3)(\bar0,\bar1,\bar3)13\underline{\bar2 1}1\bar2\bar3$
$= (0,1,3)(\bar0,\bar1,\bar3)13(0,4,3)(\bar0,\bar4,\bar3)\bar1 21\bar2\bar3 = (1,0)(3,4)(\bar1,\bar0)(\bar3,\bar4)10\bar1 2\underline{1\bar2}\bar3$
$= (1,0)(3,4)(\bar1,\bar0)(\bar3,\bar4)10\bar1 2(0,4,3)(\bar0,\bar4,\bar3)2\bar1\bar3 = (0,1,4)(\bar0,\bar1,\bar4)14\bar1 22\bar1\bar3$
$= (0,1,4)(\bar0,\bar1,\bar4)1\underline{4\bar1}\bar2\bar1\bar3 = (0,1,4)(\bar0,\bar1,\bar4)1(0,2,3)(\bar0,\bar2,\bar3)1\bar4\bar2\bar1\bar3$
$= (1,4,2,3,0)(\bar1,\bar4,\bar2,\bar3,\bar0)11\bar4\bar2\bar1\bar3 = (1,4,2,3,0)(\bar1,\bar4,\bar2,\bar3,\bar0)\bar1\bar4\bar2\bar1\bar3 \in [01204].$ $\square$

Lemma-81 ($01203\bar4 \in [01214]$)

Proof. $0120\underline{3\bar4} = 0120(2,1,0)(\bar2,\bar1,\bar0)4\bar3 = (2,1,0)(\bar2,\bar1,\bar0)2\underline{0124}\bar3$
$= (2,1,0)(\bar2,\bar1,\bar0)2(0,4,1,3,2)(\bar0,\bar4,\bar1,\bar3,\bar2)\bar2\bar1\bar0 3\bar3 = (2,3)(1,4)(\bar2,\bar3)(\bar1,\bar4)\underline{0\bar2}\bar1\bar0 3$
$= (2,3)(1,4)(\bar2,\bar3)(\bar1,\bar4)(4,1,3)(\bar4,\bar1,\bar3)2\bar0\bar1\bar0 3 = (2,4,3)(\bar2,\bar4,\bar3)2\bar0\bar1\bar0 3$
$= (2,4,3)(\bar2,\bar4,\bar3)20\underline{0\bar1}\bar0 3 = (2,4,3)(\bar2,\bar4,\bar3)20(2,4,3)(\bar2,\bar4,\bar3)1\bar0\bar0 3$
$= (2,3,4)(\bar2,\bar3,\bar4)40103 \in [01214].$ $\square$

Lemma-82 ($012\bar3\bar0 1 \in [01\bar2\bar0]$)

Proof. $012\bar3\bar0 1 = \underline{0123}\bar3\bar0 1 = (2,0,3,1,4)(\bar2,\bar0,\bar3,\bar1,\bar4)\bar2\bar1\bar0\underline{\bar4\bar3}\bar0 1$
$= (2,0,3,1,4)(\bar2,\bar0,\bar3,\bar1,\bar4)\bar2\bar1\bar0(2,1,0)(\bar2,\bar1,\bar0)\bar3 4\bar0 1 = (0,3)(1,4)(\bar0,\bar3)(\bar1,\bar4)\underline{\bar1 0\bar2\bar3}4\bar0 1$
$= (0,3)(1,4)(\bar0,\bar3)(\bar1,\bar4)(2,1,3,0,4)(\bar2,\bar1,\bar3,\bar0,\bar4)20144\bar0 1$
$= (1,2)(3,4)(\bar1,\bar2)(\bar3,\bar4)201\bar4\underline{\bar0 1} = (1,2)(3,4)(\bar1,\bar2)(\bar3,\bar4)201\bar4(4,2,3)(\bar4,\bar2,\bar3)\bar1 0$
$= (2,1,3)(\bar2,\bar1,\bar3)30\underline{1\bar2}\bar1 0 = (2,1,3)(\bar2,\bar1,\bar3)30(0,4,3)(\bar0,\bar4,\bar3)2\bar1\bar1 0$
$= (1,0,4,3,2)(\bar1,\bar0,\bar4,\bar3,\bar2)\underline{0421}0$
$= (1,0,4,3,2)(\bar1,\bar0,\bar4,\bar3,\bar2)(2,0,1,4,3)(\bar2,\bar0,\bar1,\bar4,\bar3)\bar2\bar4\bar0\bar3 0$
$= (0,3)(2,4)(\bar0,\bar3)(\bar2,\bar4)\bar2\bar4\bar0\underline{\bar3 0}t = (0,3)(2,4)(\bar0,\bar3)(\bar2,\bar4)\bar2\bar4\bar0(2,1,4)(\bar2,\bar1,\bar4)\bar0 3$
$= (0,3)(1,4)(\bar0,\bar3)(\bar1,\bar4)\bar1\bar2\bar0\bar0 3 = (0,3)(1,4)(\bar0,\bar3)(\bar1,\bar4)\bar1\underline{\bar2 0}3$
$= (0,3)(1,4)(\bar0,\bar3)(\bar1,\bar4)\bar1(4,1,3)(\bar4,\bar1,\bar3)\bar0 23 = (0,4,3)(\bar0,\bar4,\bar3)\bar3\bar0 23 \in [01\bar2\bar0].$ $\square$

Lemma-83 ($012\bar3\bar0\bar1 \in [012\bar3\bar0]$)

Proof. $012\bar3\bar0\bar1 = 01\bar2\underline{\bar2\bar3\bar0\bar1} = 01\bar2(2,1,3,4,0)(\bar2,\bar1,\bar3,\bar4,\bar0)0324$
$= (2,1,3,4,0)(\bar2,\bar1,\bar3,\bar4,\bar0)2\underline{3\bar1}0324 = (2,1,3,4,0)(\bar2,\bar1,\bar3,\bar4,\bar0)2(2,0,4)(\bar2,\bar0,\bar4)1\bar30324$
$= (2,1,3)(\bar2,\bar1,\bar3)01\underline{\bar30}324 = (2,1,3)(\bar2,\bar1,\bar3)01(2,1,4)(\bar2,\bar1,\bar4)\bar03324$
$= (1,3)(2,4)(\bar1,\bar3)(\bar2,\bar4)0\underline{4\bar0}\bar324 = (1,3)(2,4)(\bar1,\bar3)(\bar2,\bar4)0(2,1,3)(\bar2,\bar1,\bar3)0\bar4\bar324$
$= (2,4,1)(\bar2,\bar4,\bar1)\bar0\bar4\underline{\bar32}4 = (2,4,1)(\bar2,\bar4,\bar1)\bar0\bar4(0,4,1)(\bar0,\bar4,\bar1)\bar234$
$= (1,2)(0,4)(\bar1,\bar2)(\bar0,\bar4)\bar4\bar1\bar234 \in [012\bar3\bar0]$. □

Lemma-84 ($012\bar3\bar02 \in [01203]$)

Proof. $012\bar3\bar02 = \underline{0123}3\bar02 = (2,0,3,1,4)(\bar2,\bar0,\bar3,\bar1,\bar4)\bar2\bar1\bar0\underline{\bar43}\bar02$
$= (2,0,3,1,4)(\bar2,\bar0,\bar3,\bar1,\bar4)\bar2\bar1\bar0(2,1,0)(\bar2,\bar1,\bar0)\bar34\bar02 = (0,3)(1,4)(\bar0,\bar3)(\bar1,\bar4)\underline{\bar1\bar0\bar2\bar3}4\bar02$
$= (0,3)(1,4)(\bar0,\bar3)(\bar1,\bar4)(2,1,3,0,4)(\bar2,\bar1,\bar3,\bar0,\bar4)20144\bar02 = (1,2)(3,4)(\bar1,\bar2)(\bar3,\bar4)201\bar4\underline{\bar02}$
$= (1,2)(3,4)(\bar1,\bar2)(\bar3,\bar4)201\bar4(1,4,3)(\bar1,\bar4,\bar3)\bar20 = (2,4,1)(\bar2,\bar4,\bar1)204\bar3\bar20$
$= (2,4,1)(\bar2,\bar4,\bar1)\underline{2043}3\bar20 = (2,4,1)(\bar2,\bar4,\bar1)(2,3,0,1,4)(\bar2,\bar3,\bar0,\bar1,\bar4)\bar4\bar0\bar2\bar13\bar20$
$= (0,1,3)(\bar0,\bar1,\bar3)\bar4\bar0\bar2\bar1\underline{3\bar2}0 = (0,1,3)(\bar0,\bar1,\bar3)\bar4\bar0\bar2\bar1(0,1,4)(\bar0,\bar1,\bar4)2\bar30$
$= (1,3)(0,4)(\bar1,\bar3)(\bar0,\bar4)\bar0\bar1\bar2\underline{\bar42}\bar30 = (1,3)(0,4)(\bar1,\bar3)(\bar0,\bar4)\bar0\bar1\bar2(0,1,3)(\bar0,\bar1,\bar3)\bar24\bar30$
$= (0,4,1)(\bar0,\bar4,\bar1)\bar1\bar3\bar2\bar24\bar30 = (0,4,1)(\bar0,\bar4,\bar1)\bar1\underline{\bar32}4\bar30$
$= (0,4,1)(\bar0,\bar4,\bar1)\bar1(0,4,1)(\bar0,\bar4,\bar1)\bar234\bar30 = (0,1,4)(\bar0,\bar1,\bar4)\bar0\bar23\underline{4\bar3}0$
$= (0,1,4)(\bar0,\bar1,\bar4)\bar0\bar23(2,0,1)(\bar2,\bar0,\bar1)3\bar40 = (1,4)(0,2)(\bar1,\bar4)(\bar0,\bar2)\bar1\bar033\bar40$
$= (1,4)(0,2)(\bar1,\bar4)(\bar0,\bar2)\bar10\bar3\bar40 = (1,4)(0,2)(\bar1,\bar4)(\bar0,\bar2)\underline{\bar10}0\bar3\bar40$
$= (1,4)(0,2)(\bar1,\bar4)(\bar0,\bar2)(2,4,3)(\bar2,\bar4,\bar3)\bar010\bar3\bar40 = (1,3,2,0,4)(\bar1,\bar3,\bar2,\bar0,\bar4)\bar01\underline{0\bar3}\bar40$
$= (1,3,2,0,4)(\bar1,\bar3,\bar2,\bar0,\bar4)\bar01(2,1,4)(\bar2,\bar1,\bar4)3\bar0\bar40 = (1,3)(0,2)(\bar1,\bar3)(\bar0,\bar2)\bar043\bar0\underline{\bar40}$
$= (1,3)(0,2)(\bar1,\bar3)(\bar0,\bar2)\bar043\bar0(2,3,1)(\bar2,\bar3,\bar1)\bar04 = (2,0,3)(\bar2,\bar0,\bar3)\bar041\bar0\bar04$
$= (2,0,3)(\bar2,\bar0,\bar3)\underline{\bar04}104 = (2,0,3)(\bar2,\bar0,\bar3)(2,1,3)(\bar2,\bar1,\bar3)\bar40104$
$= (1,3)(0,2)(\bar1,\bar3)(\bar0,\bar2)\bar40104 = (1,3)(0,2)(\bar1,\bar3)(\bar0,\bar2)\bar4\underline{0\bar1}\bar104$
$= (1,3)(0,2)(\bar1,\bar3)(\bar0,\bar2)\bar4(2,4,3)(\bar2,\bar4,\bar3)1\bar0\bar104 = (2,0,4,3,1)(\bar2,\bar0,\bar4,\bar3,\bar1)\bar31\bar0\underline{\bar10}4$
$= (2,0,4,3,1)(\bar2,\bar0,\bar4,\bar3,\bar1)\bar31\bar0(2,4,3)(\bar2,\bar4,\bar3)\bar014 = (1,4,2,0,3)(\bar1,\bar4,\bar2,\bar0,\bar3)\bar21\bar0\bar014$
$= (1,4,2,0,3)(\bar1,\bar4,\bar2,\bar0,\bar3)\underline{\bar21}014 = (1,4,2,0,3)(\bar1,\bar4,\bar2,\bar0,\bar3)(3,0,4)(\bar3,\bar0,\bar4)\bar12014$
$= (1,3)(4,2)(\bar1,\bar3)(\bar4,\bar2)\bar1\underline{2014} = (1,3)(4,2)(\bar1,\bar3)(\bar4,\bar2)\bar1(2,4,0,3,1)(\bar2,\bar4,\bar0,\bar3,\bar1)\bar1\bar0\bar2\bar3$
$= (2,0,3)(\bar2,\bar0,\bar3)\bar2\bar1\bar0\bar2\bar3 \in [01203]$. □

Lemma-85 ($012\bar3\bar0\bar2 \in [01023]$)

Proof. $012\bar3\bar0\bar2 = \underline{0123}3\bar0\bar2 = (2,0,3,1,4)(\bar2,\bar0,\bar3,\bar1,\bar4)\bar2\bar1\bar0\underline{\bar43}\bar0\bar2$
$= (2,0,3,1,4)(\bar2,\bar0,\bar3,\bar1,\bar4)\bar2\bar1\bar0(2,1,0)(\bar2,\bar1,\bar0)\bar34\bar0\bar2 = (0,3)(1,4)(\bar0,\bar3)(\bar1,\bar4)\underline{\bar1\bar0\bar2\bar3}4\bar0\bar2$

$$= (0,3)(1,4)(\bar{0},\bar{3})(\bar{1},\bar{4})(2,1,3,0,4)(\bar{2},\bar{1},\bar{3},\bar{0},\bar{4})20144\bar{0}\bar{2} = (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})201\bar{4}\bar{0}\bar{2}$$
$$= (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})20\bar{1}\underline{\bar{1}\bar{4}\bar{0}\bar{2}} = (1,2)(3,4)(\bar{1},\bar{2})(\bar{3},\bar{4})20\bar{1}(1,2,4,3,0)(\bar{1},\bar{2},\bar{4},\bar{3},\bar{0})0413$$
$$= (1,4,0)(\bar{1},\bar{4},\bar{0})4\underline{1\bar{2}}0413 = (1,4,0)(\bar{1},\bar{4},\bar{0})4(0,4,3)(\bar{0},\bar{4},\bar{3})2\bar{1}0413$$
$$= (1,3,0)(\bar{1},\bar{3},\bar{0})32\underline{\bar{1}0}413 = (1,3,0)(\bar{1},\bar{3},\bar{0})32(2,4,3)(\bar{2},\bar{4},\bar{3})\bar{0}1413$$
$$= (1,2,4,3,0)(\bar{1},\bar{2},\bar{4},\bar{3},\bar{0})24\bar{0}1413 = (1,2,4,3,0)(\bar{1},\bar{2},\bar{4},\bar{3},\bar{0})24\bar{0}\underline{1\bar{4}}\bar{4}13$$
$$= (1,2,4,3,0)(\bar{1},\bar{2},\bar{4},\bar{3},\bar{0})24\bar{0}(2,0,3)(\bar{2},\bar{0},\bar{3})4\bar{1}\bar{4}13 = (1,0)(2,4)(\bar{1},\bar{0})(\bar{2},\bar{4})04\underline{\bar{3}4}\bar{1}\bar{4}13$$
$$= (1,0)(2,4)(\bar{1},\bar{0})(\bar{2},\bar{4})04(1,2,0)(\bar{1},\bar{2},\bar{0})\bar{4}3\bar{1}\bar{4}13 = (2,4,0)(\bar{2},\bar{4},\bar{0})14\bar{4}3\bar{1}\bar{4}13$$
$$= (2,4,0)(\bar{2},\bar{4},\bar{0})13\bar{1}\underline{\bar{4}1}3 = (2,4,0)(\bar{2},\bar{4},\bar{0})13\bar{1}(3,2,0)(\bar{3},\bar{2},\bar{0})\bar{1}43$$
$$= (2,4,3)(\bar{2},\bar{4},\bar{3})12\bar{1}\bar{1}43 = (2,4,3)(\bar{2},\bar{4},\bar{3})12143$$
$$= (2,4,3)(\bar{2},\bar{4},\bar{3})\underline{1\bar{2}}\bar{2}143 = (2,4,3)(\bar{2},\bar{4},\bar{3})(0,4,3)(\bar{0},\bar{4},\bar{3})2\bar{1}\bar{2}143$$
$$= (3,2)(0,4)(\bar{3},\bar{2})(\bar{0},\bar{4})2\bar{1}\underline{\bar{2}1}43 = (3,2)(0,4)(\bar{3},\bar{2})(\bar{0},\bar{4})2\bar{1}(0,4,3)(\bar{0},\bar{4},\bar{3})\bar{1}243$$
$$= (2,0,3)(\bar{2},\bar{0},\bar{3})2\bar{1}\bar{1}243 = (2,0,3)(\bar{2},\bar{0},\bar{3})21243 \in [01023]. \qquad \square$$

Lemma-86 (0102321 $\in$ [010232])

Proof. $0\underline{1023}21 = 0(1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2})\bar{2}\bar{0}\bar{1}\bar{4}21 = (1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2})4\bar{2}\bar{0}\bar{1}\bar{4}21$

$$= (1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2})42\underline{\bar{2}\bar{0}}\bar{1}\bar{4}21 = (1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2})42(1,4,3)(\bar{1},\bar{4},\bar{3})0\bar{2}\bar{1}\bar{4}21$$
$$= (3,0)(2,4)(\bar{3},\bar{0})(\bar{2},\bar{4})320\bar{2}\bar{1}\underline{\bar{4}2}1 = (3,0)(2,4)(\bar{3},\bar{0})(\bar{2},\bar{4})320\bar{2}\bar{1}(0,1,3)(\bar{0},\bar{1},\bar{3})\bar{2}41$$
$$= (3,1)(2,4)(\bar{3},\bar{1})(\bar{2},\bar{4})021\bar{2}\bar{3}\bar{2}41 = (3,1)(2,4)(\bar{3},\bar{1})(\bar{2},\bar{4})0212\underline{\bar{2}\bar{3}}\bar{2}41$$
$$= (3,1)(2,4)(\bar{3},\bar{1})(\bar{2},\bar{4})0212(0,4,1)(\bar{0},\bar{4},\bar{1})3\bar{2}\bar{2}41 = (1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2})4202\underline{3241}$$
$$= (1,3,0,4,2)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2})4202(4,3,1,2,0)(\bar{1},\bar{3},\bar{0},\bar{4},\bar{2})\bar{4}\bar{2}\bar{3}\bar{0} = (3,4,0)(\bar{3},\bar{4},\bar{0})304\underline{0\bar{4}}\bar{2}\bar{3}\bar{0}$$
$$= (3,4,0)(\bar{3},\bar{4},\bar{0})304(1,2,3)(\bar{1},\bar{2},\bar{3})4\bar{0}\bar{2}\bar{3}\bar{0} = (3,4,0,1,2)(\bar{3},\bar{4},\bar{0},\bar{1},\bar{2})1044\bar{0}\bar{2}\bar{3}\bar{0}$$
$$= (3,4,0,1,2)(\bar{3},\bar{4},\bar{0},\bar{1},\bar{2})1\underline{0\bar{4}}\bar{0}\bar{2}\bar{3}\bar{0} = (3,4,0,1,2)(\bar{3},\bar{4},\bar{0},\bar{1},\bar{2})1(1,2,3)(\bar{1},\bar{2},\bar{3})4\bar{0}\bar{0}\bar{2}\bar{3}\bar{0}$$
$$= (3,4,0,2,1)(\bar{3},\bar{4},\bar{0},\bar{2},\bar{1})24\underline{0\bar{2}}\bar{3}\bar{0} = (3,4,0,2,1)(\bar{3},\bar{4},\bar{0},\bar{2},\bar{1})24(3,4,1)(\bar{3},\bar{4},\bar{1})2\bar{0}\bar{3}\bar{0}$$
$$= (3,1,4,0,2)(\bar{3},\bar{1},\bar{4},\bar{0},\bar{2})212\bar{0}\bar{3}\bar{0} = (3,1,4,0,2)(\bar{3},\bar{1},\bar{4},\bar{0},\bar{2})\bar{2}\underline{\bar{2}1}2\bar{0}\bar{3}\bar{0}$$
$$= (3,1,4,0,2)(\bar{3},\bar{1},\bar{4},\bar{0},\bar{2})\bar{2}(3,0,4)(\bar{3},\bar{0},\bar{4})\bar{1}22\bar{0}\bar{3}\bar{0}$$
$$= (3,1)(2,0)(\bar{3},\bar{1})(\bar{2},\bar{0})\bar{2}\bar{1}\bar{2}\bar{0}\bar{3}\bar{0} \in [010232]. \qquad \square$$

Appendix A

Magma Scripts

Generating $A_7 \times S_3$ over S_5

```
S:=Sym(10);
xx:=S!(5,1,2,3,4)(10,6,7,8,9);
yy:=S!(5,10)(1,6)(2,7)(3,9)(4,8);
N:=sub<S|xx,yy>;
NN<x,y>:=Group<x,y|x^5,y^2,(x*y)^4,(x,y)^3>;

#N;

#NN;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..120]];
for i in [2..120] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq 1
then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1
then P[j]:=xx^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..120] do if ArrayP[i] eq N!(5,1,2,3,4)(10,6,7,8,9)
then Sch[i]; end if; end for;
```

```
G<x,y,t>:=Group<x,y,t|x^5,y^2,(x*y)^4,(x,y)^3,t^3,(y*t)^2,(y*t^x)^2,
(y*t^(x^ 2))^2,(y * x * y * x^2 * y * x^2*t^(x^2))^4>;

f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
ts:=[Id(G1) : i in [1..10]];
ts[5]:=f(t); ts[1]:=f(t^x);ts[2]:=f(t^(x^2));ts[3]:=f(t^(x^3));
ts[4]:=f(t^(x^4));
ts[10]:=(ts[5])^-1;ts[9]:=(ts[4])^-1; ts[8]:=(ts[3])^-1;
ts[6]:=(ts[1])^-1;ts[7]:=(ts[2])^-1;
cst := [null : i in [1 .. 126]] where null is [Integers() | ];
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I] applied sequentially.
*/
 v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
#G;

/* ******************************************************
                 COSET STABILIZERS
****************************************************** */

N7:=Stabiliser(N,7);
N74:=sub<N|(1,3,5)(6,8,10),(1,4,3)(6,9,8)>;
N79:=Stabiliser(N7,9);
N742:=sub<N|(1,3,5)(6,8,10),(1,4,3)(6,9,8)>;
N747:=sub<N|(1,3,5)(6,8,10),(1,4,2,5,3)(6,9,7,10,8)>;
N792:=sub<N|(1,3,5)(6,8,10),(1,5,2)(6,10,7)>;
N797:=sub<N|(1,3,5)(6,8,10),(1,3)(2,5)(6,8)(7,10)>;
N798:=sub<N|(1,6)(2,9)(3,8)(4,7)(5,10),(1,10,3,6,5,8)(2,9)(4,7)>;
N7424:=sub<N|(1,3,5)(6,8,10),(1,3,4)(6,8,9)>;
N7924:=sub<N|(1,3,5)(6,8,10),(1,8,4,6,3,9)(2,10)(5,7)>;
N7979:=sub<N|(1,3,5)(6,8,10),(1,3,5,2,4)(6,8,10,7,9)>;

/* ******************************************************
                 TRANSVERSALS
****************************************************** */
```

```
T7:=Transversal(N,N7);
T74:=Transversal(N,N74);
T79:=Transversal(N,N79);
T742:=Transversal(N,N742);
T747:=Transversal(N,N747);
T792:=Transversal(N,N792);
T797:=Transversal(N,N797);
T798:=Transversal(N,N798);
T7424:=Transversal(N,N7424);
T7924:=Transversal(N,N7924);
T7979:=Transversal(N,N7979);

/* ****************************************************
                  FIND # SINGLE COSETS
**************************************************** */

for i in [1..#T7] do
ss:=[7]^T7[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..126] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T74] do
ss:=[7,4]^T74[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..126] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T79] do
ss:=[7,9]^T79[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..126] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T742] do
ss:=[7,4,2]^T742[i];
cst[prodim(1, ts, ss)] := ss;
end for;
```

```
m:=0;
for i in [1..126] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T747] do
ss:=[7,4,7]^T747[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..126] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T792] do
ss:=[7,9,2]^T792[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..126] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T797] do
ss:=[7,9,7]^T797[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..126] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T798] do
ss:=[7,9,8]^T798[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..126] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T7424] do
ss:=[7,4,2,4]^T7424[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..126] do if cst[i] ne [] then m:=m+1; end if; end for;
m;
```

```
for i in [1..#T7924] do
ss:=[7,9,2,4]^T7924[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..126] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T7979] do
ss:=[7,9,7,9]^T7979[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..126] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

/* *******************************************************
  SYMMETRIC REPRESENTATION------>PERMUTATION REPRESENTATION
******************************************************* */

sym2per := function(G1,N, yyy)
/*
Convert an element x of J1 in the symmetric representation into a
permutation acting on 126 letters
*/
xxx := N ! yyy[1]; uuu := yyy[2];
p := [1 : i in [1 .. 126]];
ts:=[Id(G1): i in [1..10]];
ts[5]:=f(t); ts[1]:=f(t^x);ts[2]:=f(t^(x^2));ts[3]:=f(t^(x^3));
ts[4]:=f(t^(x^4));ts[10]:=(ts[5])^-1;ts[9]:=(ts[4])^-1;
ts[8]:=(ts[3])^-1;ts[6]:=(ts[1])^-1;ts [7]:=(ts[2])^-1;
p := [1 : i in [1 .. 126]];
for i := 1 to 10 do
p[prodim(1, ts, [i])] := prodim(1, ts, [i]^xxx);;
end for;

T7:=Transversal(N,N7);
for i := 1 to #T7 do
ss:=[7]^T7[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T74:=Transversal(N,N74);
for i := 1 to #T74 do
```

```
ss:=[7,4]^T74[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T79:=Transversal(N,N79);
for i := 1 to #T79 do
ss:=[7,9]^T79[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T742:=Transversal(N,N742);
for i := 1 to #T742 do
ss:=[7,4,2]^T742[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T747:=Transversal(N,N747);
for i := 1 to #T747 do
ss:=[7,4,7]^T747[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T792:=Transversal(N,N792);
for i := 1 to #T792 do
ss:=[7,9,2]^T792[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T797:=Transversal(N,N797);
for i := 1 to #T797 do
ss:=[7,9,7]^T797[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T798:=Transversal(N,N798);
for i := 1 to #T798 do
ss:=[7,9,8]^T798[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T7424:=Transversal(N,N7424);
for i := 1 to #T7424 do
ss:=[7,4,2,4]^T7424[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
```

```
end for;

T7924:=Transversal(N,N7924);
for i := 1 to #T7924 do
ss:=[7,9,2,4]^T7924[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T7979:=Transversal(N,N7979);
for i := 1 to #T7979 do
ss:=[7,9,7,9]^T7979[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

return (G1 ! p) * &*[G1|ts[uuu[j]]: j in [1 .. #uuu]];
end function;

/* ************************************************************
      PERMUTATION REPRESENTATION---->SYMMETRIC REPRESENTATION
************************************************************ */

per2sym := function(G1,N, p)
/*
Convert permutation p of J1 on 126 letters into its symmetric
representation
*/
ww := cst[1^p];
tt := p * &*[G1|(ts[ww[#ww - l + 1]])^-1: l in [1 .. #ww]];
zz := N![rep{j: j in [1..10] | (1^ts[i])^tt eq 1^ts[j]}: i in [1..10]];
return <zz, ww>;
end function;
```

Generating $3{\cdot}A_7$ over S_4

```
S:=Sym(6);
xx:=S!(3,1,6,4)(2,5);
yy:=S!(1,5)(4,2);
N:=sub<S|xx,yy>;
NN<x,y>:=Group<x,y|x^4,y^2,(x*y)^3>;
#N;

#NN;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..24]];
for i in [2..24] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq 1
then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1
then P[j]:=xx^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..24] do if ArrayP[i] eq N!(1,5)(4,2)
then Sch[i]; end if; end for;

G<x,y,t>:=Group<x,y,t|x^4,y^2,(y*x)^3,t^3,(t,y),(x^2*t)^2,(y*t^x)^5>;

#G;

#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);

f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
ts:=[Id(G1) : i in [1..6]];
ts[3]:=f(t); ts[1]:=f(t^x);ts[2]:=f((t^(x^3))^y);
ts[6]:=(ts[3])^-1;ts[5]:=(ts[2])^-1; ts[4]:=(ts[1])^-1;
cst := [null : i in [1 .. 315]] where null is [Integers() | ];
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I] applied sequentially
```

```
*/
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
#G;

/* *******************************************************
                    COSET STABILIZERS
******************************************************* */

N3:=Stabiliser(N,3);
N31:=Stabiliser(N3,1);
N312:=Stabiliser(N31,2);
N316:=sub<N|(1,3)(6,4),(1,4)(3,6)>;
N315:=Stabiliser(N31,5);
N3123:=Stabiliser(N312,3);
N3126:=Stabiliser(N312,6);
N3124:=sub<N|(1,5)(2,4),(1,2)(4,5)>;
N3162:=sub<N|(1,3)(6,4),(1,4)(3,6)>;
N3165:=sub<N|(1,3)(6,4),(1,6)(3,4)>;
N3153:=Stabiliser(N315,3);
N3154:=sub<N|(1,2)(4,5),(1,4)(2,5)>;
N31543:=sub<N|(1,5)(2,4),(1,2)(4,5)>;
N31546:=sub<N|(1,2)(4,5),(1,4)(2,5)>;
N31531:=Stabiliser(N3153,1);
N31246:=sub<N|(1,2)(4,5),(1,4)(2,5)>;
N31243:=sub<N|(1,5)(2,4),(1,2)(4,5)>;
N31261:=Stabiliser(N3126,1);
N31234:=Stabiliser(N3123,4);
N31231:=Stabiliser(N3123,1);
N312312:=sub<N|(1,5)(2,4),(1,2)(4,5)>;
N315315:=sub<N|(1,4)(2,5),(1,2)(4,5)>;
N3123123:=sub<N|(1,4)(2,3,5,6),(1,3,5)(2,4,6)>;
N3153153:=sub<N|(1,4)(2,3,5,6),(1,2,4,5)(3,6)>;

/* *******************************************************
                    TRANSVERSALS
******************************************************* */

T3:=Transversal(N,N3);
T31:=Transversal(N,N31);
```

```
T312:=Transversal(N,N312);
T316:=Transversal(N,N316);
T315:=Transversal(N,N315);
T3123:=Transversal(N,N3123);
T3126:=Transversal(N,N3126);
T3124:=Transversal(N,N3124);
T3162:=Transversal(N,N3162);
T3165:=Transversal(N,N3165);
T3153:=Transversal(N,N3153);
T3154:=Transversal(N,N3154);
T31543:=Transversal(N,N31543);
T31546:=Transversal(N,N31546);
T31531:=Transversal(N,N31531);
T31246:=Transversal(N,N31246);
T31243:=Transversal(N,N31243);
T31261:=Transversal(N,N31261);
T31234:=Transversal(N,N31234);
T31231:=Transversal(N,N31231);
T312312:=Transversal(N,N312312);
T315315:=Transversal(N,N315315);
T3123123:=Transversal(N,N3123123);
T3153153:=Transversal(N,N3153153);

/* ****************************************************
                   FIND # SINGLE COSETS
**************************************************** */

for i in [1..#T3] do
ss:=[3]^T3[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T31] do
ss:=[3,1]^T31[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T312] do
```

```
ss:=[3,1,2]^T312[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T316] do
ss:=[3,1,6]^T316[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T315] do
ss:=[3,1,5]^T315[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T3123] do
ss:=[3,1,2,3]^T3123[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T3126] do
ss:=[3,1,2,6]^T3126[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T3124] do
ss:=[3,1,2,4]^T3124[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
```

```
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T3162] do
ss:=[3,1,6,2]^T3162[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T3165] do
ss:=[3,1,6,5]^T3165[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T3153] do
ss:=[3,1,5,3]^T3153[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T3154] do
ss:=[3,1,5,4]^T3154[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T31543] do
ss:=[3,1,5,4,3]^T31543[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T31546] do
```

```
ss:=[3,1,5,4,6]^T31546[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T31531] do
ss:=[3,1,5,3,1]^T31531[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T31246] do
ss:=[3,1,2,4,6]^T31246[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T31243] do
ss:=[3,1,2,4,3]^T31243[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T31261] do
ss:=[3,1,2,6,1]^T31261[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T31234] do
ss:=[3,1,2,3,4]^T31234[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
```

```
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T31231] do
ss:=[3,1,2,3,1]^T31231[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T312312] do
ss:=[3,1,2,3,1,2]^T312312[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T315315] do
ss:=[3,1,5,3,1,5]^T315315[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T3153153] do
ss:=[3,1,5,3,1,5,3]^T3153153[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T3123123] do
ss:=[3,1,2,3,1,2,3]^T3123123[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

/* *************************************************************
```

```
     SYMMETRIC REPRESENTATION---->PERMUTATION REPRESENTATION
************************************************************ */

sym2per := function(G1,N, yyy)
/*
Convert an element x of J1 in the symmetric representation into a
permutation acting on 315 letters
*/
xxx := N ! yyy[1]; uuu := yyy[2];
p := [1 : i in [1 .. 315]];
ts:=[Id(G1): i in [1..6]];
ts[3]:=f(t); ts[1]:=f(t^x);ts[2]:=f((t^(x^3))^y);
ts[6]:=(ts[3])^-1;ts[5]:=(ts[2])^-1; ts[4]:=(ts[1])^-1;
p := [1 : i in [1 .. 315]];
for i := 1 to 6 do
p[prodim(1, ts, [i])] := prodim(1, ts, [i]^xxx);;
end for;

T3:=Transversal(N,N3);
for i := 1 to #T3 do
ss:=[3]^T3[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T31:=Transversal(N,N31);
for i := 1 to #T31 do
ss:=[3,1]^T31[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T312:=Transversal(N,N312);
for i := 1 to #T312 do
ss:=[3,1,2]^T312[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T316:=Transversal(N,N316);
for i := 1 to #T316 do
ss:=[3,1,6]^T316[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T315:=Transversal(N,N315);
for i := 1 to #T315 do
```

```
ss:=[3,1,5]^T315[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T3123:=Transversal(N,N3123);
for i := 1 to #T3123 do
ss:=[3,1,2,3]^T3123[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T3126:=Transversal(N,N3126);
for i := 1 to #T3126 do
ss:=[3,1,2,6]^T3126[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T3124:=Transversal(N,N3124);
for i := 1 to #T3124 do
ss:=[3,1,2,4]^T3124[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T3162:=Transversal(N,N3162);
for i := 1 to #T3162 do
ss:=[3,1,6,2]^T3162[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T3165:=Transversal(N,N3165);
for i := 1 to #T3165 do
ss:=[3,1,6,5]^T3165[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T3153:=Transversal(N,N3153);
for i := 1 to #T3153 do
ss:=[3,1,5,3]^T3153[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T3154:=Transversal(N,N3154);
for i := 1 to #T3154 do
ss:=[3,1,5,4]^T3154[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
```

```
end for;

T31543:=Transversal(N,N31543);
for i := 1 to #T31543 do
ss:=[3,1,5,4,3]^T31543[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T31546:=Transversal(N,N31546);
for i := 1 to #T31546 do
ss:=[3,1,5,4,6]^T31546[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T31531:=Transversal(N,N31531);
for i := 1 to #T31531 do
ss:=[3,1,5,3,1]^T31531[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T31246:=Transversal(N,N31246);
for i := 1 to #T31246 do
ss:=[3,1,2,4,6]^T31246[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T31243:=Transversal(N,N31243);
for i := 1 to #T31243 do
ss:=[3,1,2,4,3]^T31243[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T31261:=Transversal(N,N31261);
for i := 1 to #T31261 do
ss:=[3,1,2,6,1]^T31261[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T31234:=Transversal(N,N31234);
for i := 1 to #T31234 do
ss:=[3,1,2,3,4]^T31234[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;
```

```
T31231:=Transversal(N,N31231);
for i := 1 to #T31231 do
ss:=[3,1,2,3,1]^T31231[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T312312:=Transversal(N,N312312);
for i := 1 to #T312312 do
ss:=[3,1,2,3,1,2]^T312312[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;


T315315:=Transversal(N,N315315);
for i := 1 to #T315315 do
ss:=[3,1,5,3,1,5]^T315315[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T3123123:=Transversal(N,N3123123);
for i := 1 to #T3123123 do
ss:=[3,1,2,3,1,2,3]^T3123123[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T3153153:=Transversal(N,N3153153);
for i := 1 to #T3153153 do
ss:=[3,1,5,3,1,5,3]^T3153153[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

return (G1 ! p) * &*[G1|ts[uuu[j]]: j in [1 .. #uuu]];
end function;

/* ************************************************************
   PERMUTATION REPRESENTATION---->SYMMETRIC REPRESENTATION
************************************************************ */

per2sym := function(G1,N, p)
/*
Convert permutation p of J1 on 315 letters into its symmetric
representation
*/
ww := cst[1^p];
```

```
tt := p * &*[G1|(ts[ww[#ww - l + 1]])^-1: l in [1 .. #ww]];
zz := N![rep{j: j in [1..6] | (1^ts[i])^tt eq 1^ts[j]}: i in [1..6]];
return <zz, ww>;
end function;
```

Generating A_7 over S_4

```
S:=Sym(6);
xx:=S!(3,1,6,4)(2,5);
yy:=S!(1,5)(4,2);
N:=sub<S|xx,yy>;
NN<x,y>:=Group<x,y|x^4,y^2,(x*y)^3>;
#N;

#NN;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..24]];
for i in [2..24] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq 1
then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1
then P[j]:=xx^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..24] do if ArrayP[i] eq N!(1,3,2)(4,6,5)
then Sch[i]; end if; end for;

G<x,y,t>:=Group<x,y,t|x^4,y^2,(y*x)^3,t^3,(t,y),(x^2*t)^2,(y*t^x)^5,
x^-1*y*t*t^x*(t^(x^3))^y*t*t^x*(t^(x^3))^y*t>;

#G;

#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);

f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
ts:=[Id(G1) : i in [1..6]];
ts[3]:=f(t); ts[1]:=f(t^x);ts[2]:=f((t^(x^3))^y);
ts[6]:=(ts[3])^-1;ts[5]:=(ts[2])^-1; ts[4]:=(ts[1])^-1;
cst := [null : i in [1 .. 105]] where null is [Integers() | ];
prodim := function(pt, Q, I)
/*
```

```
Return the image of pt under permutations Q[I] applied sequentially
*/
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
#G;

/* ***********************************************************
                    COSET STABILIZERS
*********************************************************** */

N3:=Stabiliser(N,3);
N31:=Stabiliser(N3,1);
N312:=Stabiliser(N31,2);
N316:=sub<N|(1,3)(6,4),(1,4)(3,6)>;
N315:=Stabiliser(N31,5);
N3126:=sub<N|(1,2,6)(3,4,5),(1,6,2)(3,5,4)>;
N3124:=sub<N|(1,4)(2,5),(1,2)(4,5)>;
N3165:=sub<N|(1,3)(6,4),(1,6)(3,4)>;

/* ***********************************************************
                     TRANSVERSALS
*********************************************************** */

T3:=Transversal(N,N3);
T31:=Transversal(N,N31);
T312:=Transversal(N,N312);
T316:=Transversal(N,N316);
T315:=Transversal(N,N315);
T3126:=Transversal(N,N3126);
T3124:=Transversal(N,N3124);
T3165:=Transversal(N,N3165);

/* ***********************************************************
                    FIND # SINGLE COSETS
*********************************************************** */

for i in [1..#T3] do
ss:=[3]^T3[i];
cst[prodim(1, ts, ss)] := ss;
end for;
```

```
m:=0;
for i in [1..105] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T31] do
ss:=[3,1]^T31[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..105] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T312] do
ss:=[3,1,2]^T312[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..105] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T316] do
ss:=[3,1,6]^T316[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..105] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T315] do
ss:=[3,1,5]^T315[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..105] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T3126] do
ss:=[3,1,2,6]^T3126[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..105] do if cst[i] ne [] then m:=m+1; end if; end for;
m;
```

```
for i in [1..#T3124] do
ss:=[3,1,2,4]^T3124[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..105] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T3165] do
ss:=[3,1,6,5]^T3165[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..105] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

/* ***********************************************************
    SYMMETRIC REPRESENTATION---->PERMUTATION REPRESENTATION
*********************************************************** */

sym2per := function(G1,N, yyy)
/*
Convert an element x of J1 in the symmetric repesentation into a
permutation acting on 105 letters
*/
xxx := N ! yyy[1]; uuu := yyy[2];
p := [1 : i in [1 .. 105]];
ts:=[Id(G1): i in [1..6]];
ts[3]:=f(t); ts[1]:=f(t^x);ts[2]:=f((t^(x^3))^y);
ts[6]:=(ts[3])^-1;ts[5]:=(ts[2])^-1; ts[4]:=(ts[1])^-1;
p := [1 : i in [1 .. 105]];
for i := 1 to 6 do
p[prodim(1, ts, [i])] := prodim(1, ts, [i]^xxx);;
end for;

T3:=Transversal(N,N3);
for i := 1 to #T3 do
ss:=[3]^T3[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T31:=Transversal(N,N31);
for i := 1 to #T31 do
ss:=[3,1]^T31[i];
```

```
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T312:=Transversal(N,N312);
for i := 1 to #T312 do
ss:=[3,1,2]^T312[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T316:=Transversal(N,N316);
for i := 1 to #T316 do
ss:=[3,1,6]^T316[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T315:=Transversal(N,N315);
for i := 1 to #T315 do
ss:=[3,1,5]^T315[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T3126:=Transversal(N,N3126);
for i := 1 to #T3126 do
ss:=[3,1,2,6]^T3126[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

fT3124:=Transversal(N,N3124);
for i := 1 to #T3124 do
ss:=[3,1,2,4]^T3124[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T3165:=Transversal(N,N3165);
for i := 1 to #T3165 do
ss:=[3,1,6,5]^T3165[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

return (G1 ! p) * &*[G1|ts[uuu[j]]: j in [1 .. #uuu]];
end function;

/* *******************************************************
   PERMUTATION REPRESENTATION---->SYMMETRIC REPRESENTATION
```

```
******************************************************** */

per2sym := function(G1,N, p)
/*
Convert permutation p of J1 on 105 letters into its symmetric
representation
*/
ww := cst[1^p];
tt := p * &*[G1!(ts[ww[#ww - l + 1]])^-1: l in [1 .. #ww]];
zz := N![rep{j: j in [1..6] | (1^ts[i])^tt eq 1^ts[j]}: i in [1..6]];
return <zz, ww>;
end function;
```

Generating $3^{\cdot}A_7$ over $L_2(7) \times 3$

```
S:=Sym(6);
xx:=S!(3,1,6,4)(2,5);
yy:=S!(1,5)(4,2);
N:=sub<S|xx,yy>;
NN<x,y>:=Group<x,y|x^4,y^2,(x*y)^3>;
#N;

#NN;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..24]];
for i in [2..24] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq 1
then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1
then P[j]:=xx^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..24] do if ArrayP[i] eq N!(3,2)(6,5)
then Sch[i]; end if; end for;

G<x,y,t>:=Group<x,y,t|x^4,y^2,(y*x)^3,t^3,(t,y),(x^2*t)^2,(y*t^x)^5>;

#G;

H:=sub<G|x,y,x*y*x^-1*t*((t^(x^3))^y)^-1*t^-1*(t^x)^-1>;

#H;

#DoubleCosets(G,sub<G|x,y,x*y*x^-1*t*((t^(x^3))^y)^-1*t^-1*(t^x)^-1>,
sub<G|x,y>);

f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
IH:=sub<G1|f(x),f(y),f(x*y*x^-1*t*((t^(x^3))^y)^-1*t^-1*(t^x)^-1)>;
ts:=[Id(G1) : i in [1..6]];
```

```
ts[3]:=f(t); ts[1]:=f(t^x);ts[2]:=f((t^(x^3))^y);
ts[6]:=(ts[3])^-1;ts[5]:=(ts[2])^-1; ts[4]:=(ts[1])^-1;
cst := [null : i in [1 .. 315]] where null is [Integers() | ];
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I] applied sequentially
*/
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
#G;

H3:=Stabiliser(N,3);
H31:=sub<N|(1,5,3)(2,6,4),(1,3,5)(2,4,6)>;


T3:=Transversal(N,H3);
T31:=Transversal(N,H31);

for i in [1..#T3] do
ss:=[3]^T3[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;


for i in [1..#T31] do
ss:=[3,1]^T31[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..315] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

sym2per := function(G1,N, yyy)
/*
Convert an element x of J1 in the symmetric representation into a
permutation acting on 315 letters
*/
```

```
xxx := N ! yyy[1]; uuu := yyy[2];
p := [1 : i in [1 .. 15]];
ts:=[Id(G1): i in [1..6]];
ts[3]:=f(t); ts[1]:=f(t^x);ts[2]:=f((t^(x^3))^y);
ts[6]:=(ts[3])^-1;ts[5]:=(ts[2])^-1; ts[4]:=(ts[1])^-1;
p := [1 : i in [1 .. 15]];
for i := 1 to 6 do
p[prodim(1, ts, [i])] := prodim(1, ts, [i]^xxx);;
end for;

T3:=Transversal(N,H3);
for i := 1 to #T3 do
ss:=[3]^T3[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T31:=Transversal(N,H31);
for i := 1 to #T31 do
ss:=[3,1]^T31[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

return (G1 ! p) * &*[G1|ts[uuu[j]]: j in [1 .. #uuu]];
end function;
```

Generating $M_{12} \times 2$ over S_5

```
S:=Sym(10);
xx:=S!(5,1,2,3,4)(10,6,7,8,9);
yy:=S!(5,10)(1,6)(2,7)(3,9)(4,8);
N:=sub<S|xx,yy>;
NN<x,y>:=Group<x,y|x^5,y^2,(x*y)^4,(x,y)^3>;

#N;

#NN;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..120]];
for i in [2..120] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq 1
then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1
then P[j]:=xx^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for i in [1..120] do if ArrayP[i] eq N!(5,3,1,4,2)(10,8,6,9,7)
then Sch[i]; end if; end for;

G<x,y,t>:=Group<x,y,t|x^5,y^2,(x*y)^4,(x,y)^3,t^3,(y*t)^2,(y*t^x)^2,
(y*t^(x^2))^2,(y * x * y * x^2 * y* x^2*t)^4,(x*t)^8>;

#G;

#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);

f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
ts:=[Id(G1) : i in [1..10]];
ts[5]:=f(t); ts[1]:=f(t^x);ts[2]:=f(t^(x^2));ts[3]:=f(t^(x^3));
ts[4]:=f(t^(x^4));ts[10]:=(ts[5])^-1;ts[9]:=(ts[4])^-1;
ts[8]:=(ts[3])^-1;ts[6]:=(ts[1])^-1;ts[7]:=(ts[2])^-1;
cst := [null : i in [1 .. 1584]] where null is [Integers() | ];
```

```
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I] applied sequentially.
*/
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
#G;

/* *******************************************************
                        COSET STABILIZERS
******************************************************* */

N5:=Stabiliser(N,5);
N51:=sub<N|(2,3,4)(7,8,9)>;
N56:=sub<N|(2,3,4)(7,8,9),(1,5)(2,3)(6,10)(7,8)>;
N515:=sub<N|(1,5)(2,3)(6,10)(7,8),(1,6)(2,7)(3,9)(4,8)(5,10)>;
N512:=Stabiliser(N51,2);
N517:=sub<N|(1,2)(4,5)(6,7)(9,10)>;
N562:=sub<N|(1,2,5,4,3)(6,7,10,9,8),(1,2)(3,5)(6,7)(8,10)>;
N567:=sub<N|(1,5)(3,4)(6,10)(8,9)>;
N5152:=sub<N|(1,5)(3,4)(6,10)(8,9)>;
N5125:=Stabiliser(N512,5);
N5121:=sub<N|(1,2)(3,5)(6,7)(8,10)>;
N5123:=Stabiliser(N512,3);
N5128:=sub<N|(1,3)(2,4)(6,8)(7,9)>;
N51710:=sub<N|(1,8)(2,7)(3,6)(4,9)(5,10)>;
N5178:=sub<N|(1,10,2,9)(3,8)(4,6,5,7),(1,9,2,10)(3,8)(4,7,5,6)>;
N5673:=sub<N|(2,3)(4,5)(7,8)(9,10)>;
N5678:=sub<N|(1,5)(2,3)(6,10)(7,8)>;
N51523:=Stabiliser(N5152,3);
N51251:=sub<N|(1,9)(2,7)(3,8)(4,6)(5,10)>;
N51254:=sub<N|(1,2)(4,5)(6,7)(9,10)>;
N51258:=Stabiliser(N5125,8);
N51214:=sub<N|(1,2)(3,5)(6,7)(8,10)>;
N51253:=Stabiliser(N5152,3);
N512810:=sub<N|(1,3)(2,4)(8,6)(7,9)>;
N56784:=sub<N|(2,5)(3,4)(7,10)(8,9),(1,8,5,7)(2,6,3,10)(4,9)>;
N515232:=sub<N|(1,6)(2,8)(3,7)(4,9)(5,10),(2,4)(3,5)(7,9)(8,10)>;

/* *******************************************************
```

```
                        TRANSVERSAL
****************************************************** */

T5:=Transversal(N,N5);
T51:=Transversal(N,N51);
T56:=Transversal(N,N56);
T515:=Transversal(N,N515);
T512:=Transversal(N,N512);
T517:=Transversal(N,N517);
T562:=Transversal(N,N562);
T567:=Transversal(N,N567);
T5152:=Transversal(N,N5152);
T5125:=Transversal(N,N5125);
T5121:=Transversal(N,N5121);
T5123:=Transversal(N,N5123);
T5128:=Transversal(N,N5128);
T51710:=Transversal(N,N51710);
T5178:=Transversal(N,N5178);
T5673:=Transversal(N,N5673);
T5678:=Transversal(N,N5678);
T51523:=Transversal(N,N51523);
T51251:=Transversal(N,N51251);
T51254:=Transversal(N,N51254);
T51258:=Transversal(N,N51258);
T51214:=Transversal(N,N51214);
T51253:=Transversal(N,N51253);
T512810:=Transversal(N,N512810);
T56784:=Transversal(N,N56784);
T515232:=Transversal(N,N515232);

/* ******************************************************
                  FIND # SINGLE COSETS
****************************************************** */

for i in [1..#T5] do
ss:=[5]^T5[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T51] do
ss:=[5,1]^T51[i];
```

```
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T56] do
ss:=[5,6]^T56[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T515] do
ss:=[5,1,5]^T515[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T512] do
ss:=[5,1,2]^T512[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T517] do
ss:=[5,1,7]^T517[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T562] do
ss:=[5,6,2]^T562[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
```

```
m;

for i in [1..#T567] do
ss:=[5,6,7]^T567[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T5152] do
ss:=[5,1,5,2]^T5152[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T5125] do
ss:=[5,1,2,5]^T5125[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T5121] do
ss:=[5,1,2,1]^T5121[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T5123] do
ss:=[5,1,2,3]^T5123[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T5128] do
ss:=[5,1,2,8]^T5128[i];
```

```
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;


for i in [1..#T51710] do
ss:=[5,1,7,10]^T51710[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T5178] do
ss:=[5,1,7,8]^T5178[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T5673] do
ss:=[5,6,7,3]^T5673[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T5678] do
ss:=[5,6,7,8]^T5678[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T51523] do
ss:=[5,1,5,2,3]^T51523[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
```

```
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T51251] do
ss:=[5,1,2,5,1]^T51251[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T51254] do
ss:=[5,1,2,5,4]^T51254[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T51258] do
ss:=[5,1,2,5,8]^T51258[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T51214] do
ss:=[5,1,2,1,4]^T51214[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T51253] do
ss:=[5,1,2,5,3]^T51253[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T512810] do
```

```
ss:=[5,1,2,8,10]^T512810[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T56784] do
ss:=[5,6,7,8,4]^T56784[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

for i in [1..#T515232] do
ss:=[5,1,5,2,3,2]^T515232[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..1584] do if cst[i] ne [] then m:=m+1; end if; end for;
m;

/* *********************************************************
  SYMMETRIC REPRESENTATION------>PERMUTATION REPRESENTATION
********************************************************* */

sym2per := function(G1,N, yyy)
/*
Convert an element x of J1 in the symmetric repesentation into a
permutation acting on 1584 letters.
*/
xxx := N ! yyy[1]; uuu := yyy[2];
p := [1 : i in [1 .. 1584]];
ts:=[Id(G1): i in [1..10]];
ts[5]:=f(t); ts[1]:=f(t^x);ts[2]:=f(t^(x^2));ts[3]:=f(t^(x^3));
ts[4]:=f(t^(x^4));ts[10]:=(ts[5])^-1;ts[9]:=(ts[4])^-1;
ts[8]:=(ts[3])^-1;ts[6]:=(ts[1])^-1;ts [7]:=(ts[2])^-1;
p := [1 : i in [1 .. 1584]];
for i := 1 to 10 do
p[prodim(1, ts, [i])] := prodim(1, ts, [i]^xxx);;
end for;

T5:=Transversal(N,N5);
```

```
for i := 1 to #T5 do
ss:=[5]^T5[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T51:=Transversal(N,N51);
for i := 1 to #T51 do
ss:=[5,1]^T51[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T56:=Transversal(N,N56);
for i := 1 to #T56 do
ss:=[5,6]^T56[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T515:=Transversal(N,N515);
for i := 1 to #T515 do
ss:=[5,1,5]^T515[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T512:=Transversal(N,N512);
for i := 1 to #T512 do
ss:=[5,1,2]^T512[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T517:=Transversal(N,N517);
for i := 1 to #T517 do
ss:=[5,1,7]^T517[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T562:=Transversal(N,N562);
for i := 1 to #T562 do
ss:=[5,6,2]^T562[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T567:=Transversal(N,N567);
for i := 1 to #T567 do
ss:=[5,6,7]^T567[i];
```

```
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T5152:=Transversal(N,N5152);
for i := 1 to #T5152 do
ss:=[5,1,5,2]^T5152[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T5125:=Transversal(N,N5125);
for i := 1 to #T5125 do
ss:=[5,1,2,5]^T5125[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T5121:=Transversal(N,N5121);
for i := 1 to #T5121 do
ss:=[5,1,2,1]^T5121[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T5123:=Transversal(N,N5123);
for i := 1 to #T5123 do
ss:=[5,1,2,3]^T5123[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T5128:=Transversal(N,N5128);
for i := 1 to #T5128 do
ss:=[5,1,2,8]^T5128[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T51710:=Transversal(N,N51710);
for i := 1 to #T51710 do
ss:=[5,1,7,10]^T51710[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T5178:=Transversal(N,N5178);
for i := 1 to #T5178 do
ss:=[5,1,7,8]^T5178[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;
```

```
T5673:=Transversal(N,N5673);
for i := 1 to #T5673 do
ss:=[5,6,7,3]^T5673[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T5678:=Transversal(N,N5678);
for i := 1 to #T5678 do
ss:=[5,6,7,8]^T5678[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T51523:=Transversal(N,N51523);
for i := 1 to #T51523 do
ss:=[5,1,5,2,3]^T51523[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T51251:=Transversal(N,N51251);
for i := 1 to #T51251 do
ss:=[5,1,2,5,1]^T51251[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T51254:=Transversal(N,N51254);
for i := 1 to #T51254 do
ss:=[5,1,2,5,4]^T51254[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T51258:=Transversal(N,N51258);
for i := 1 to #T51258 do
ss:=[5,1,2,5,8]^T51258[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T51214:=Transversal(N,N51214);
for i := 1 to #T51214 do
ss:=[5,1,2,1,4]^T51214[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T51253:=Transversal(N,N51253);
```

```
for i := 1 to #T51253 do
ss:=[5,1,2,5,3]^T51253[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T512810:=Transversal(N,N512810);
for i := 1 to #T512810 do
ss:=[5,1,2,8,10]^T512810[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T56784:=Transversal(N,N56784);
for i := 1 to #T56784 do
ss:=[5,6,7,8,4]^T56784[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

T515232:=Transversal(N,N515232);
for i := 1 to #T515232 do
ss:=[5,1,5,2,3,2]^T515232[i];
p[prodim(1, ts, ss)] := prodim(1, ts, ss^xxx);
end for;

return (G1 ! p) * &*[G1|ts[uuu[j]]: j in [1 .. #uuu]];
end function;

/* ********************************************************
  PERMUTATION REPRESENTATION---->SYMMETRIC REPRESENTATION
******************************************************** */

per2sym := function(G1,N, p)
/*
Convert permutation p of J1 on 1584 letters into its symmetric
representation
*/
ww := cst[1^p];
tt := p * &*[G1|(ts[ww[#ww - l + 1]])^-1: l in [1 .. #ww]];
zz := N![rep{j: j in [1..10] | (1^ts[i])^tt eq 1^ts[j]}: i in [1..10]];
return <zz, ww>;
end function;
```

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